Matching of asymptotic expansions for the wave propagation in media with thin slot

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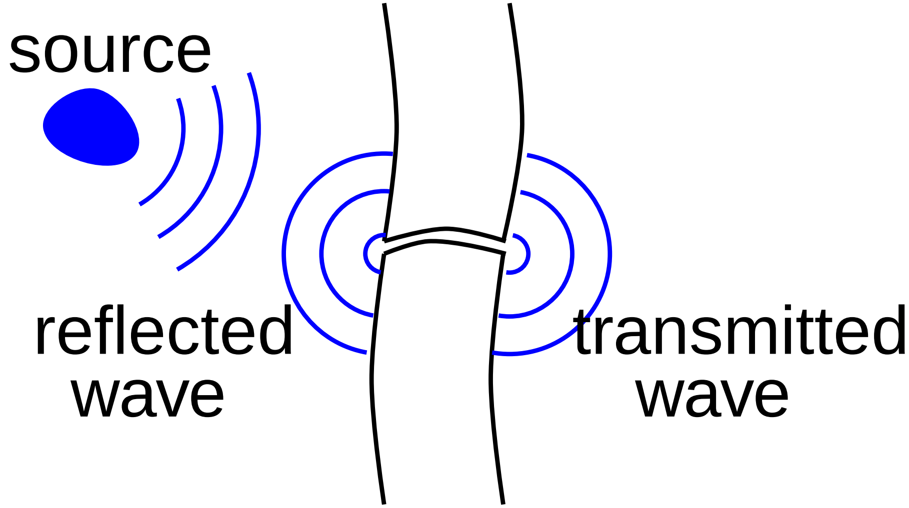
A typical application

How can we study the scattering in media with thin slot ?

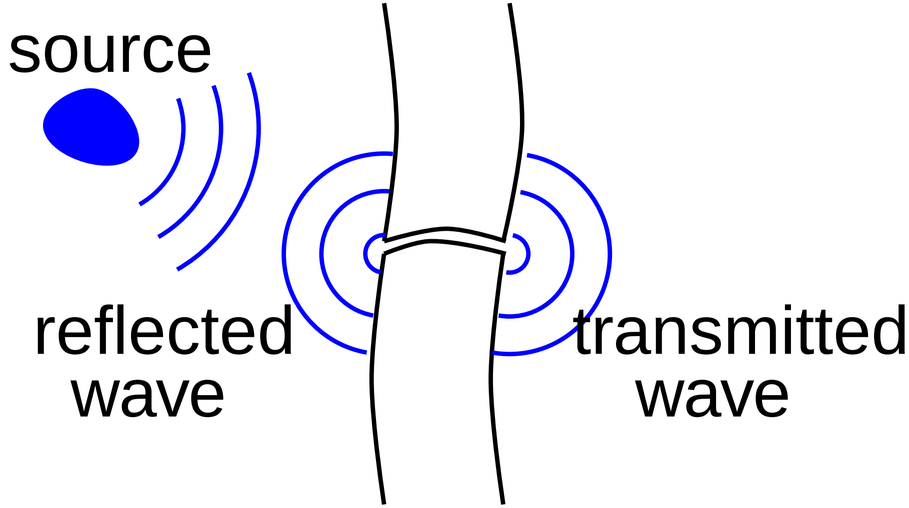


A physical problem with two caracteristical lengthes

$$\begin{cases} \text{The wavelength } \lambda \\ \text{The width of the slot } \varepsilon \end{cases}$$

A typical application

How can we study the scattering in media with thin slot ?

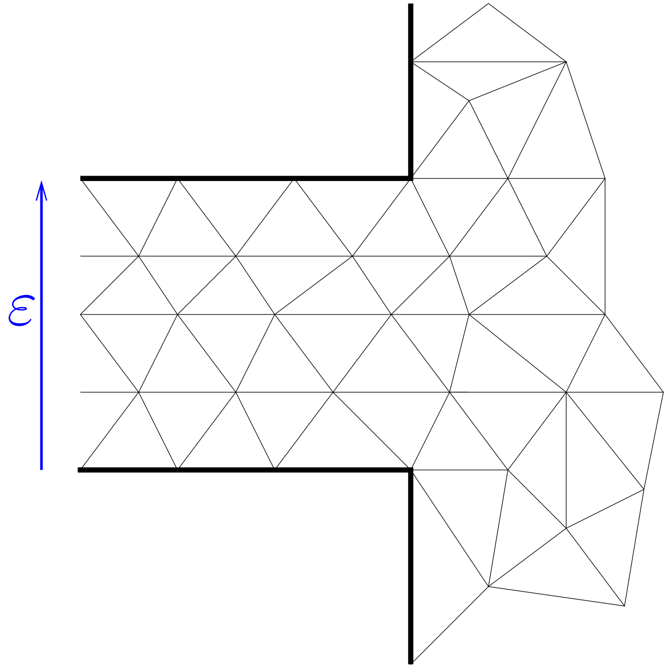


An asymptotic case:

$$\varepsilon \ll \lambda$$

The numerical difficulty

A mesh step smaller than ε



This leads to costly computations

Some references

- Thin slot:

  Harrington, Auckland (1980), Tatout (1996).

- Finite differences:

  Taflove (1995).

- Thin plates and junction theory,...

  Ciarlet, Le Dret, Dauge-Costabel.

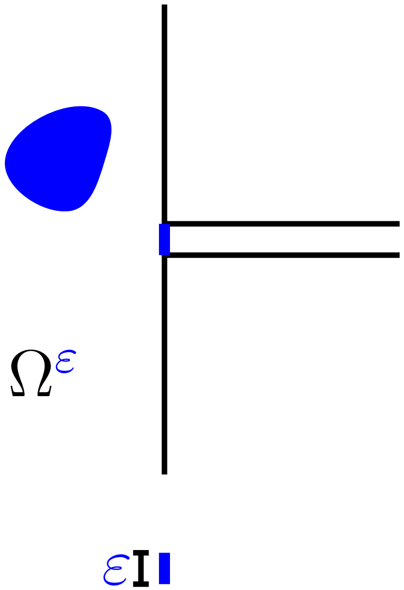
- Matching of asymptotic expansions:

  McIver, Rawlins (1993), Il'in (1992).

- multiscale analysis

  Maz'ya, Nazarov, Plamenevskii (1991)
  Oleinik, Shamaev, Yosifian (1992)

A simple problem



Scalar wave equation:

$$\frac{\partial^2 p^\varepsilon}{\partial t^2} - \Delta p^\varepsilon = f$$

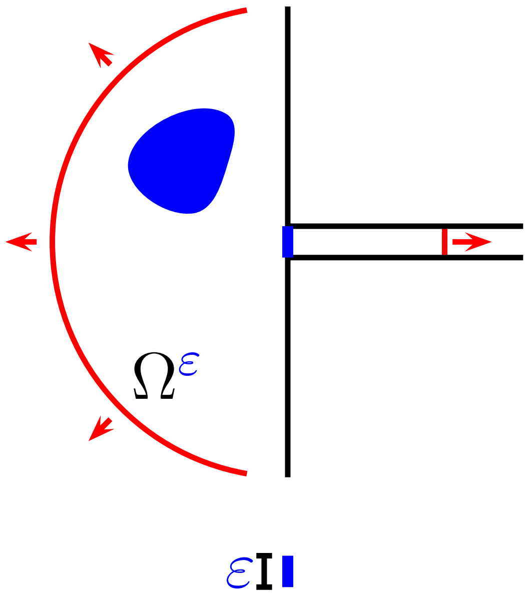
Harmonic solution:

$$p^\varepsilon(x,y,t) = exp(-\mathbf{i}\omega t)\ u^\varepsilon(x,y)$$

Helmholtz Equation:

$$\Delta u^\varepsilon + \omega^2 u^\varepsilon = -f \quad \text{in } \Omega^\varepsilon$$

A simple problem



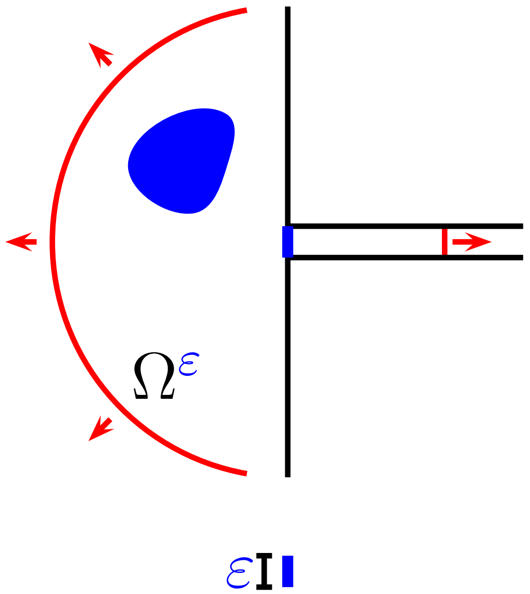
Outgoing solution at infinity:

$$\frac{\partial u^{\varepsilon}}{\partial n} - \mathbf{i}\omega u^{\varepsilon} \leq \frac{C}{r^2}, \qquad \text{for } r \text{ large,}$$

Neumann limit condition
(rigid wall)

$$\frac{\partial u^{\varepsilon}}{\partial n} = 0 \quad \text{on } \partial\Omega^{\varepsilon}$$

A simple problem



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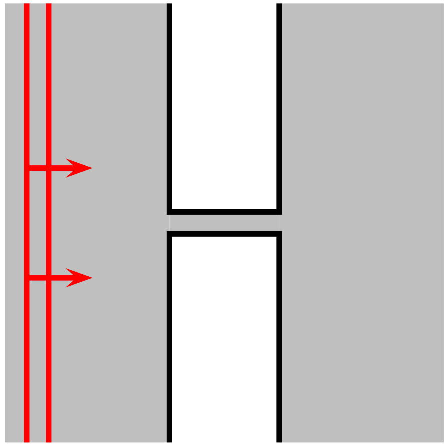
With the Dirichlet limit condition, the transmission inside the slot is negligible ($o(\varepsilon^{\infty})$).

A numerical computation

Neumann

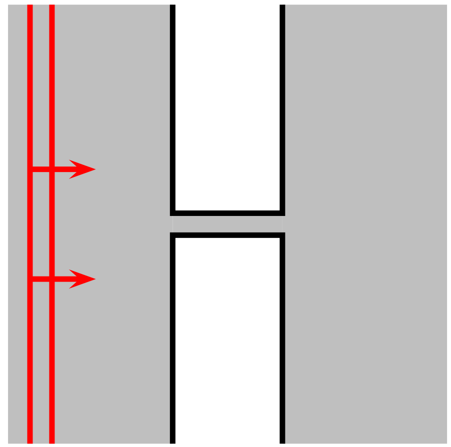


Dirichlet

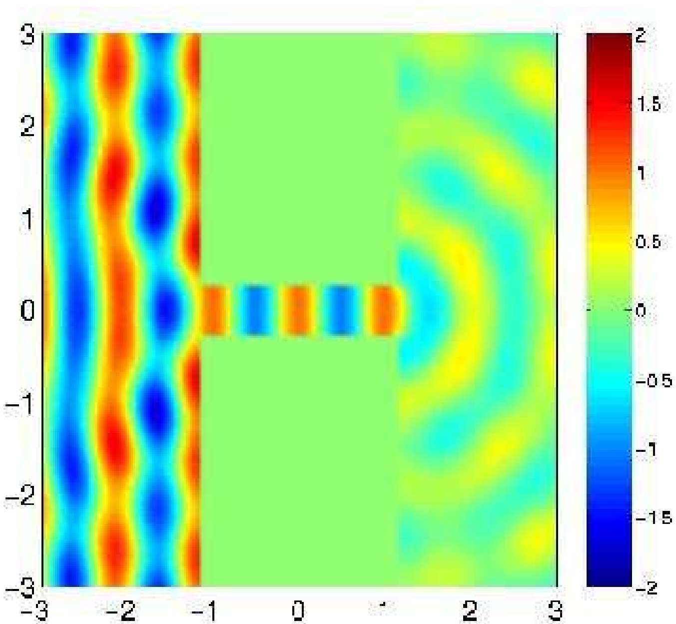
Numerical computation done with the high order finite elements code of (M. Duruflé, INRIA)

A numerical computation

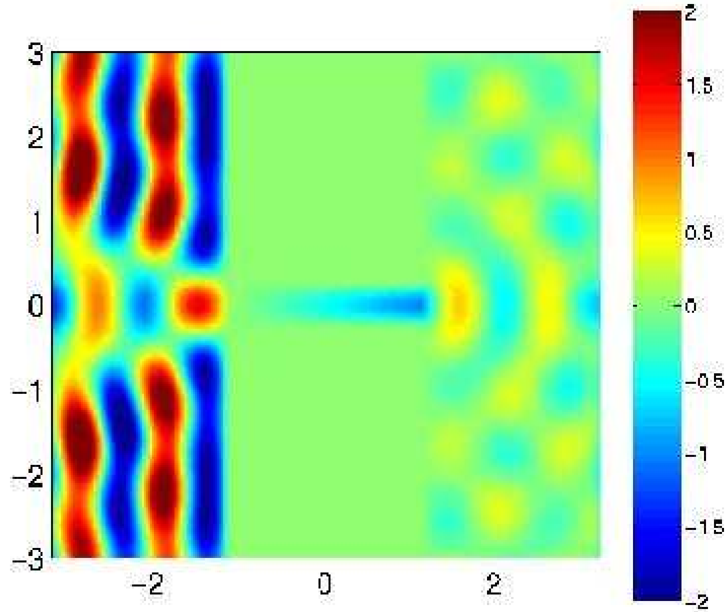
Neumann



Dirichlet

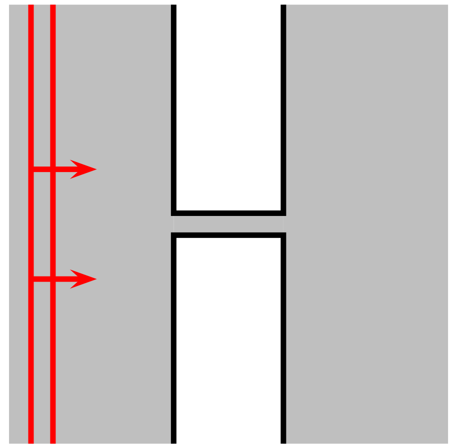


$$\frac{\varepsilon}{\lambda} = 0.5$$

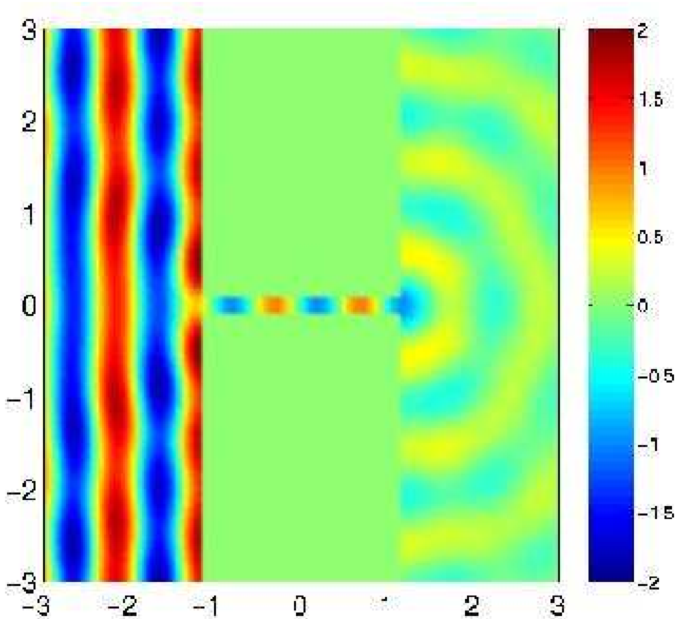


A numerical computation

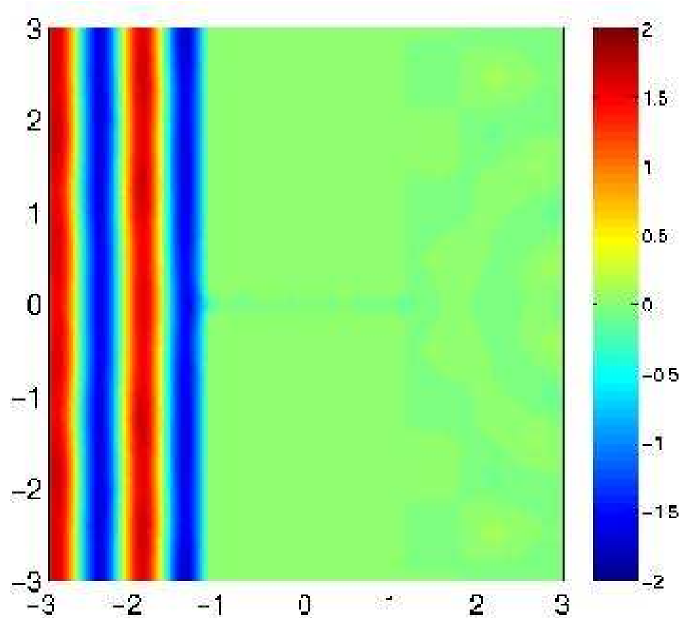
Neumann



Dirichlet

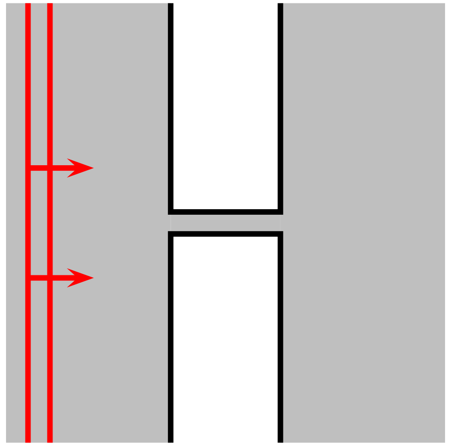


$$\frac{\varepsilon}{\lambda} = 0.2$$



A numerical computation

Neumann



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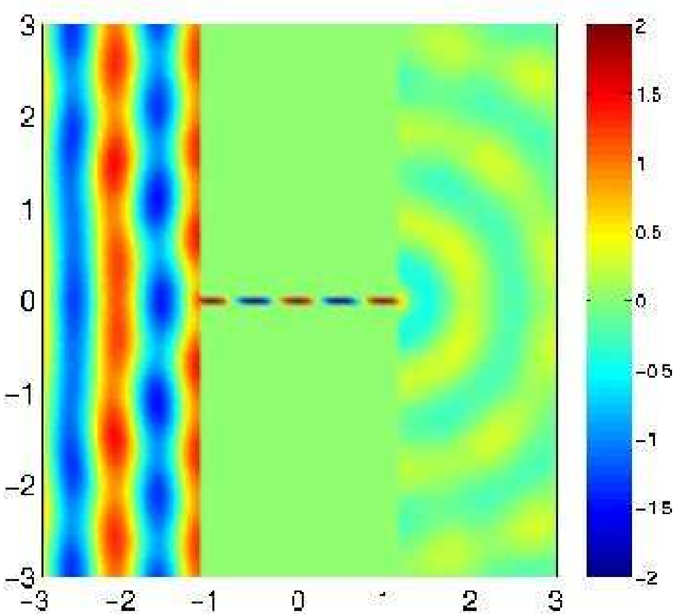
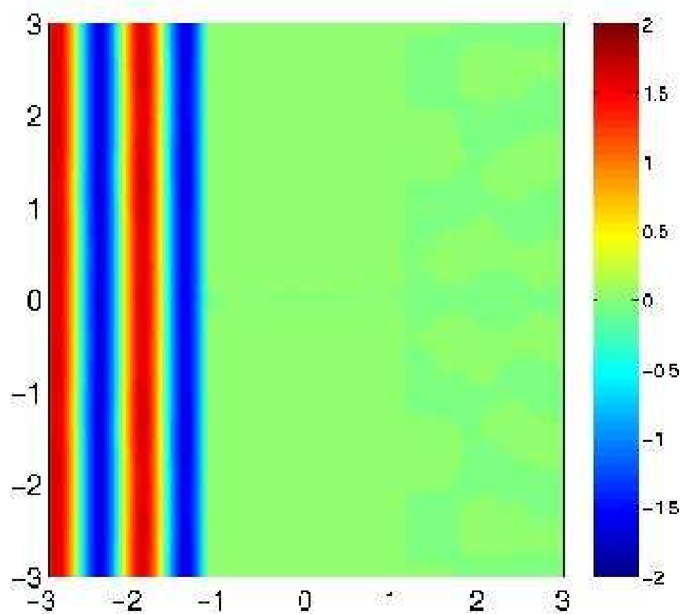


$$\frac{\textcolor{blue}{\varepsilon}}{\lambda} = 0.1$$



Objectives

- Introduce accurate numerical methods

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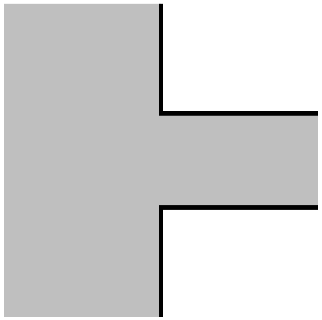
- We need an intermediate zone



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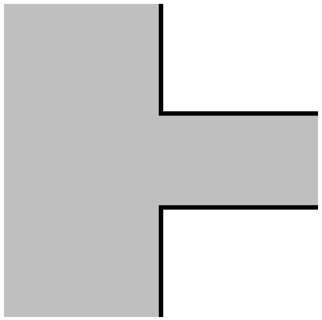

- A technique the matching of asymptotic expansions
  - Define new approximate models to compute the solution.
  - Use effectively "universal" technique of numerical computation (mesh reffinement).

Contributions to the match. of as. exp.

- A new presentation of the matching principle (not allways clear) postulated by the english school.

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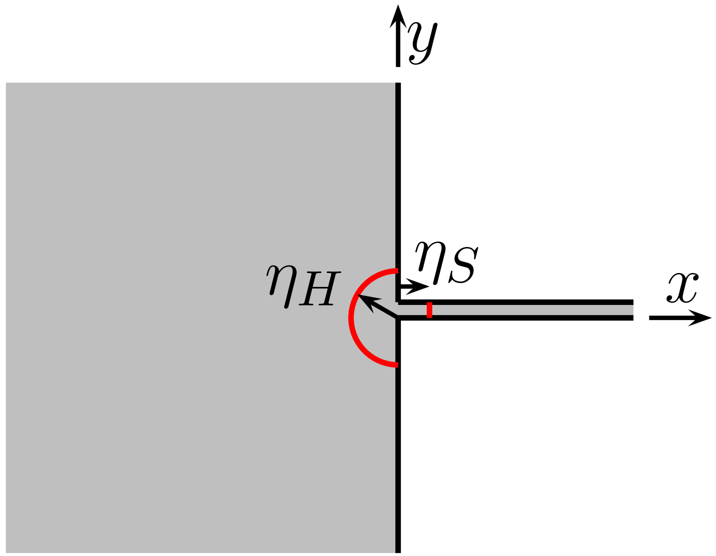
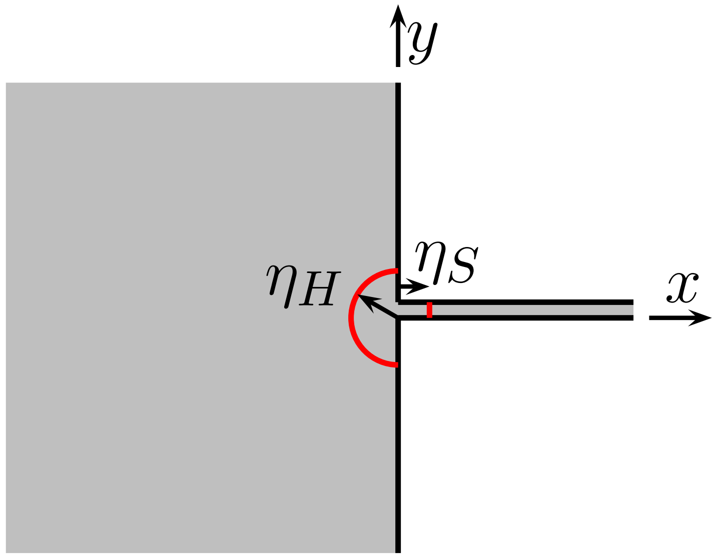
- A new presentation of the matching principle (not allways clear) postulated by the english school.
- The mathematical justification of this technique.
  - The proof are inspirated by the multiscale technique
  - Existence and unicity of the terms of the expansions.
  - Specific technique: error estimates.

Three zones



- Far field (2D field)
- Near field (boundary layer)
- Slot field (1D field)

Three zones

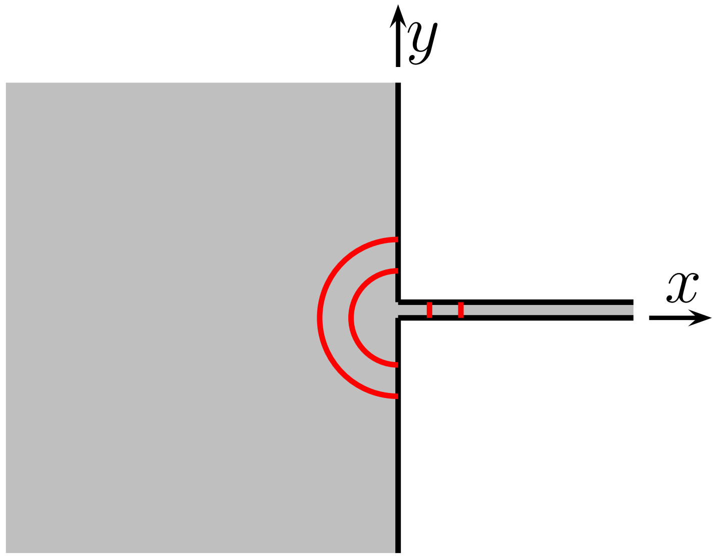


The asymptotic assumptions:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \qquad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \longrightarrow 0 \qquad \eta(\varepsilon) \longrightarrow 0 \qquad \eta(\varepsilon)/\varepsilon \longrightarrow +\infty$$

Three zones



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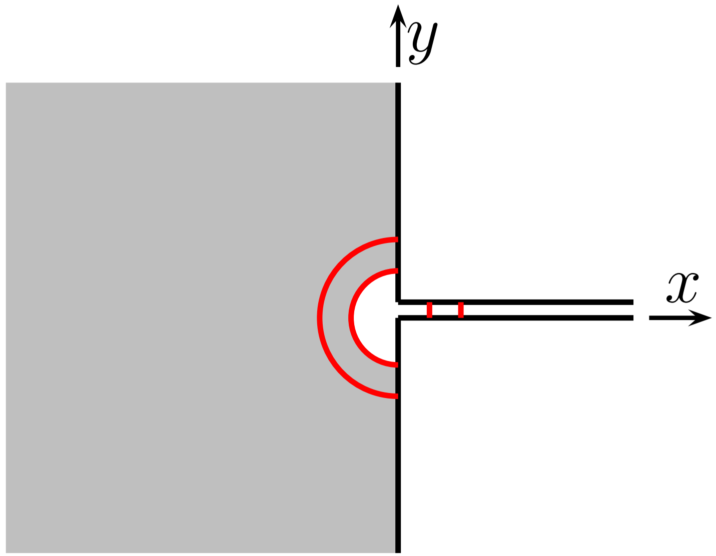

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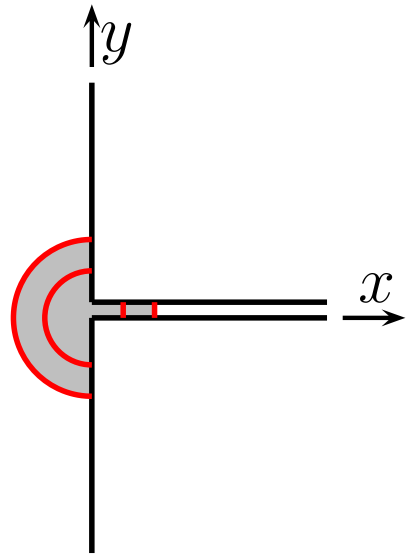

Near field

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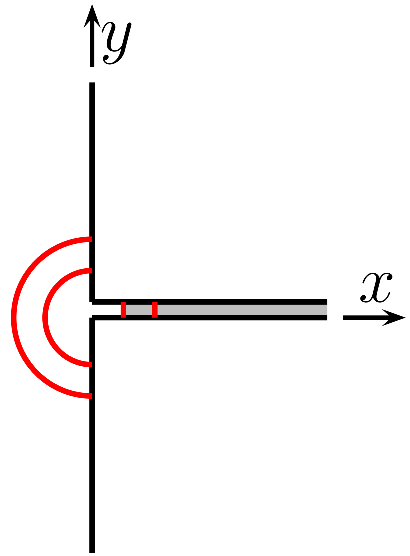

Slot field

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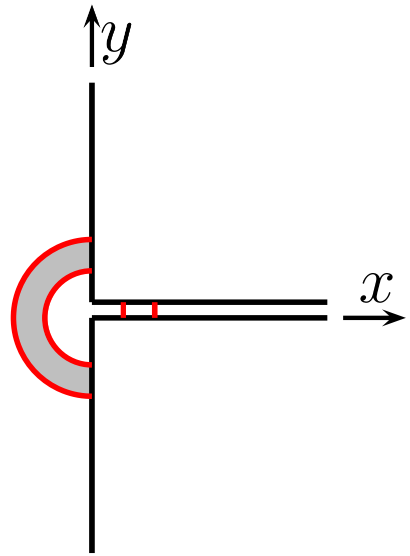

Far and near

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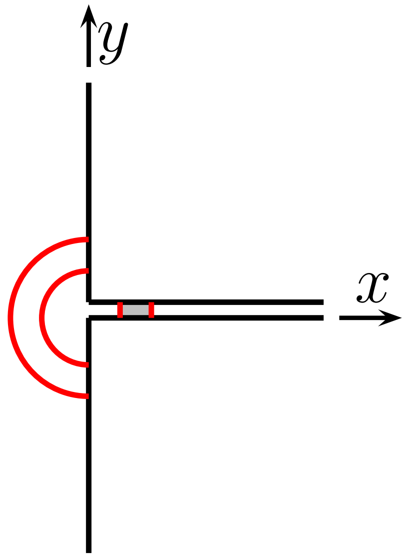

Slot and near

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The different steps of the method

- Derivate the asymptotic expansions:
    - Formal part
    - Several presentations are possible

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- Mathematical validation of the asymptotic expansions
  - Rigorous part
  - Error estimates

The different steps of the method

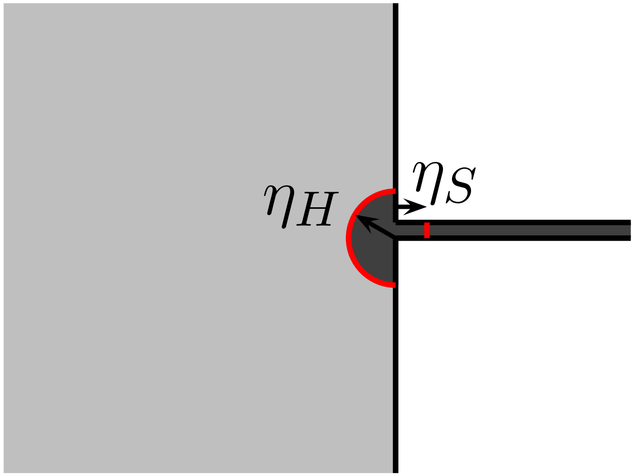
2 Derivate the asymptotic expansions:
- Formal part
- Several presentations are possible

1 Describe the asymptotic expansions
- Rigorous part
- Definition of the terms of the asymptotic expansions

3 Mathematical validation of the asymptotic expansions
- Rigorous part
- Error estimates

Far field

Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda$.



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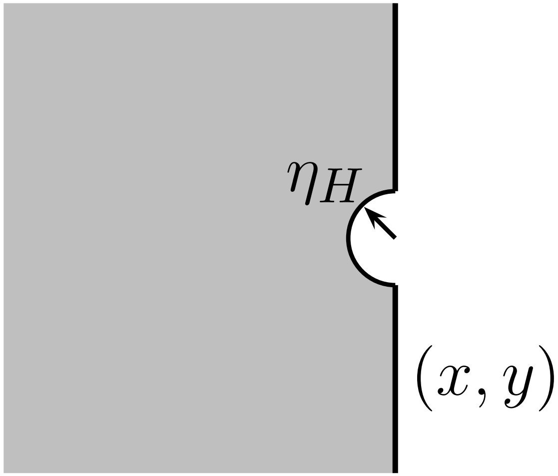

No normalization:

$$X = x, \qquad Y = y.$$

Far field

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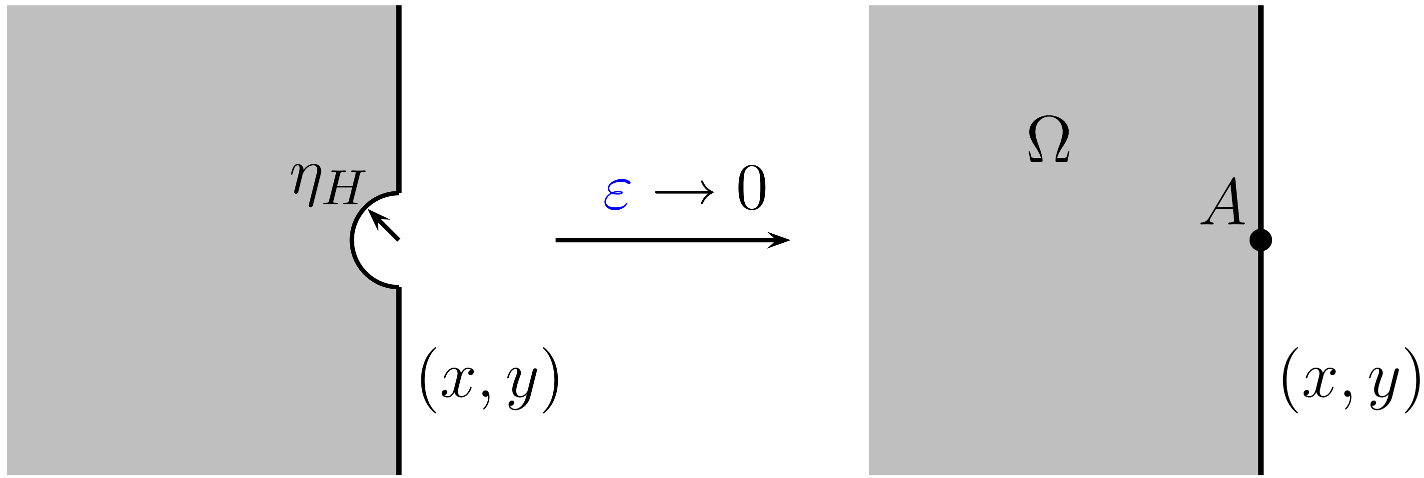

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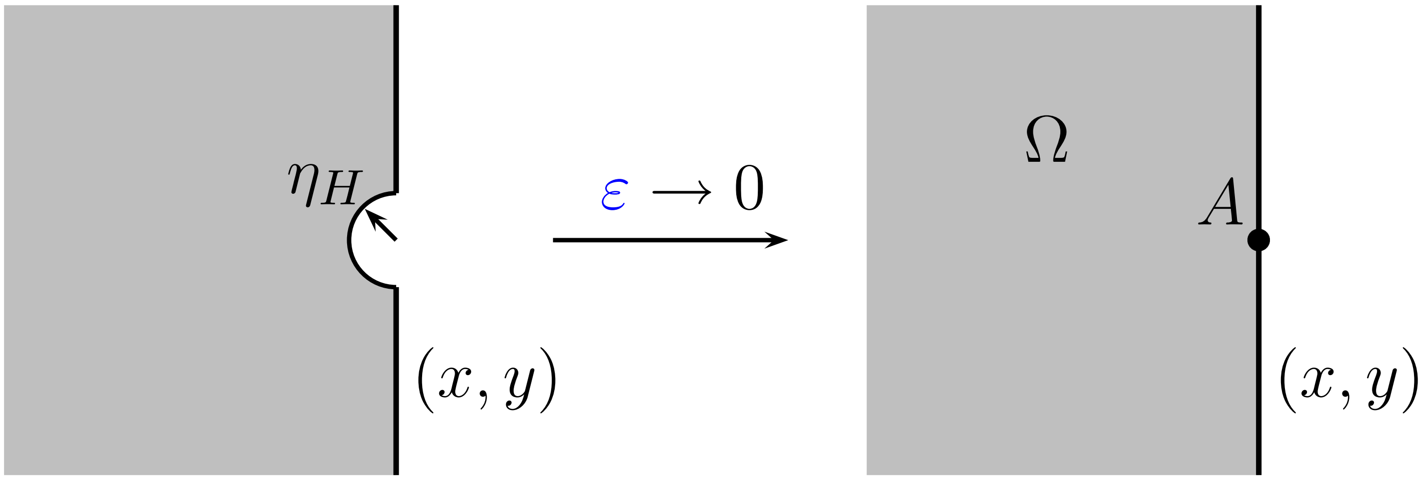
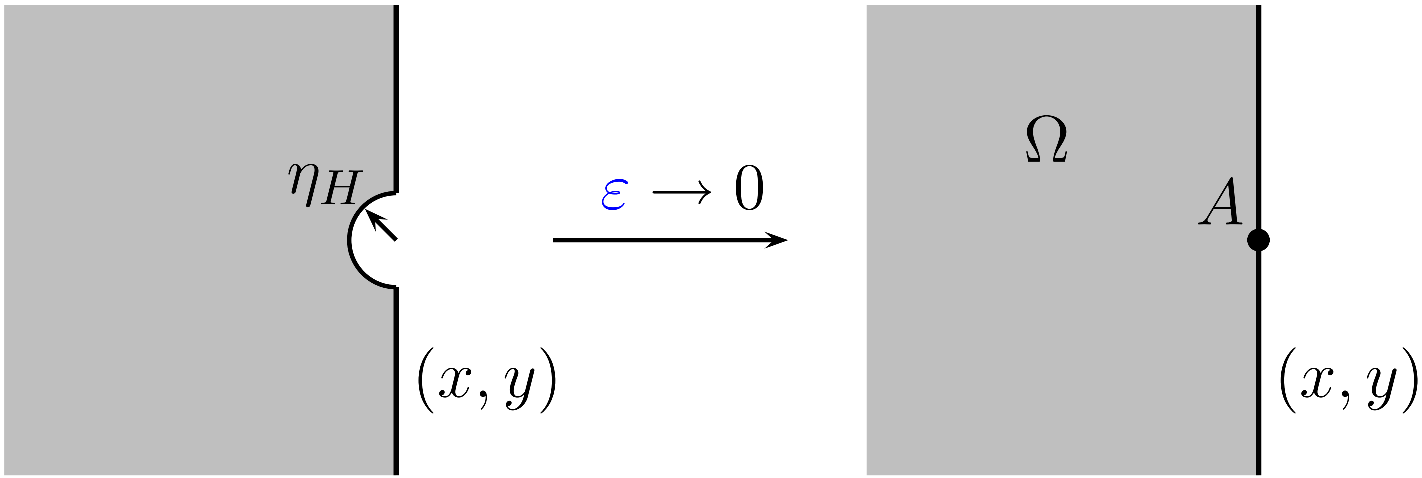


$$u^\varepsilon = u^0 + \sum_{i=1}^{+\infty} \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k + o(\varepsilon^\infty), \qquad \text{in } \Omega.$$

Far field

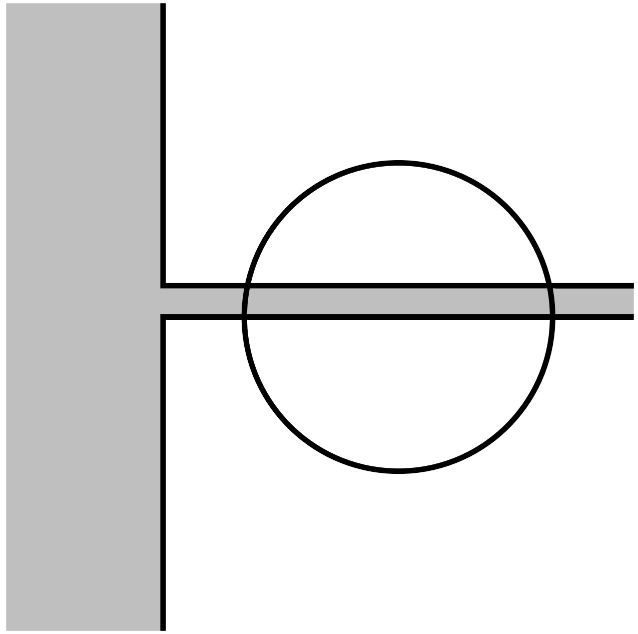
Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda$.



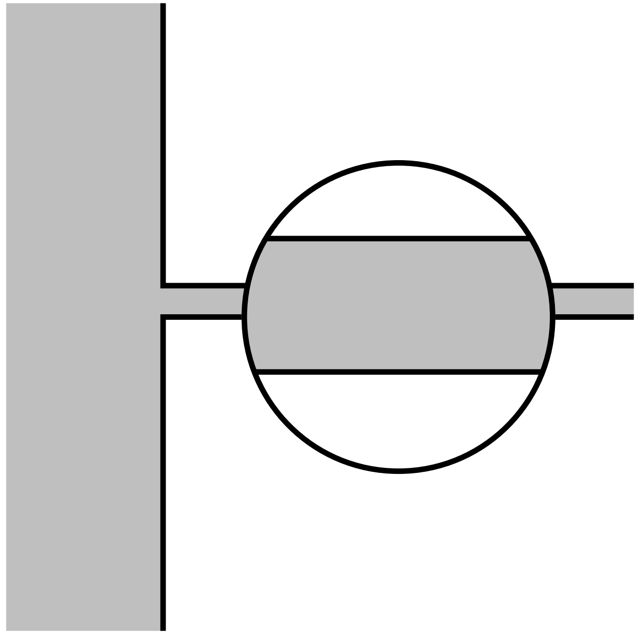
where the u_i^k satisfy the homogeneous Helmholtz equation

$$\Delta u_i^k + \omega^2 u_i^k = 0.$$

Slot field

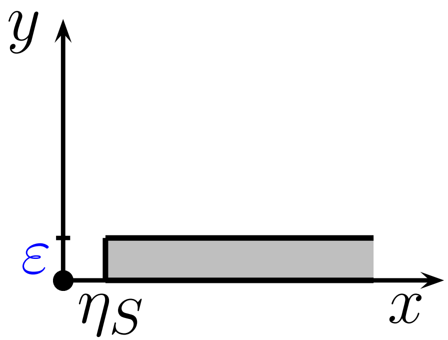


Slot field



$$u^{\varepsilon}(x,y) \;=\; U^{\varepsilon}(x,\frac{y}{\varepsilon})$$

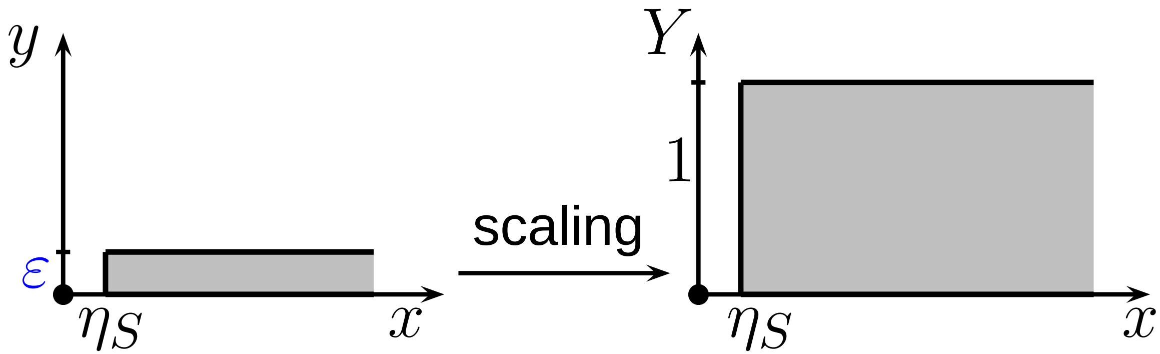
Slot field



The asymptotic context: $\varepsilon \ll \eta_S \ll \lambda$.

The normalization: $X = x, \quad Y = \dfrac{y}{\varepsilon}$

Slot field



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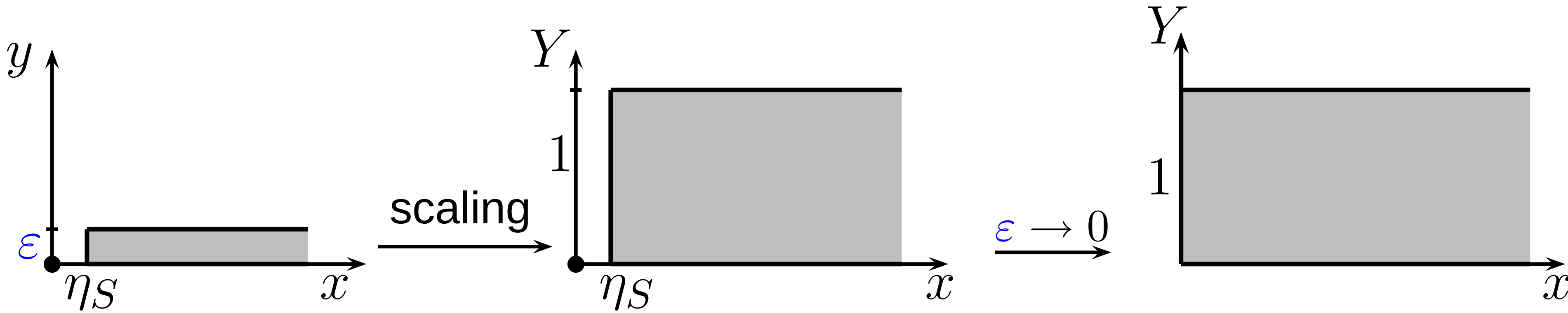

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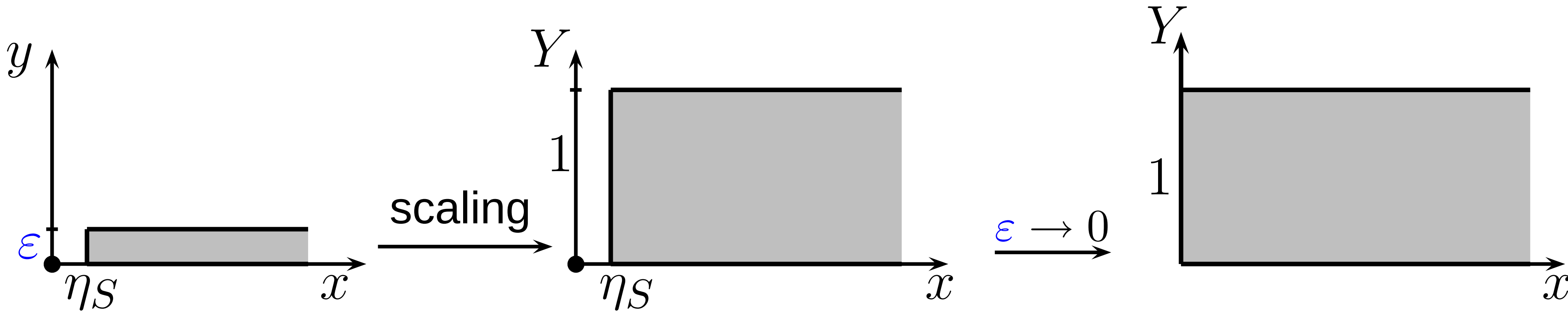
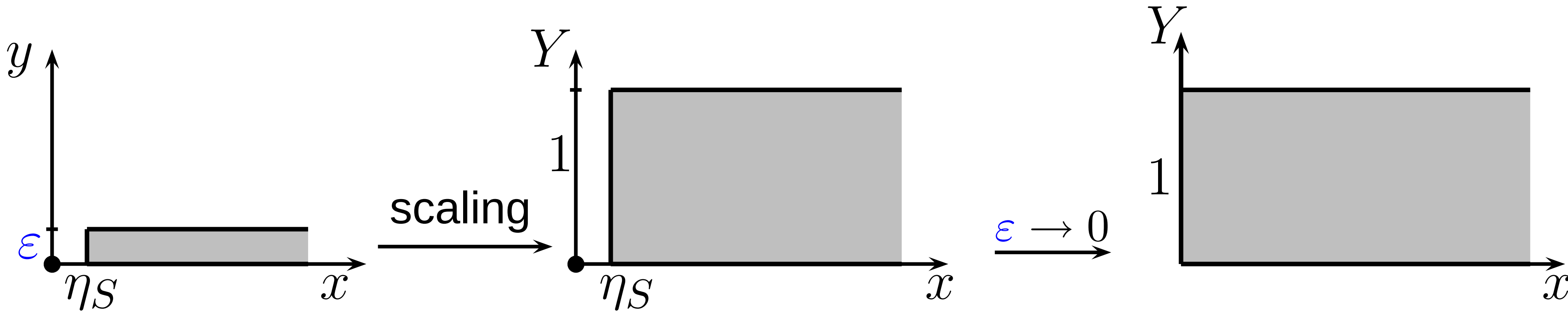


$$u^{\varepsilon}(x, Y\varepsilon) = U^{\varepsilon}(x, Y) = \sum_{i=0}^{+\infty} \sum_{k=0}^{i} \varepsilon^{i} \, (\log \varepsilon)^{k} \, U_i^k(x, Y) + o(\varepsilon^{\infty}),$$

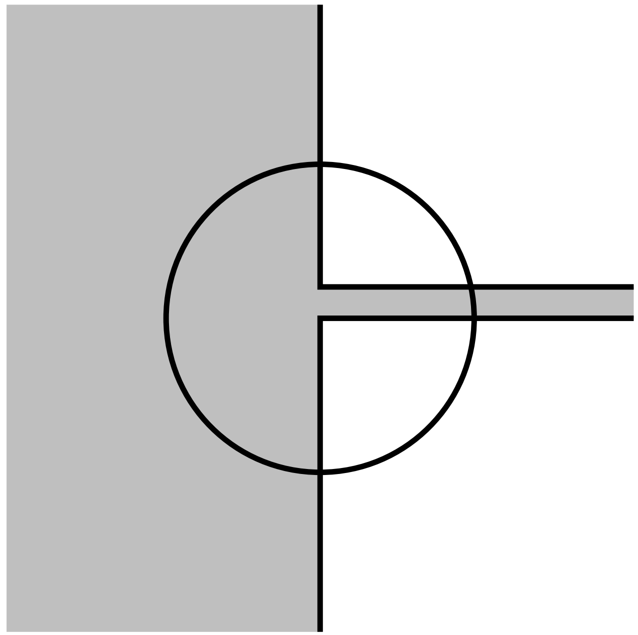
Slot field



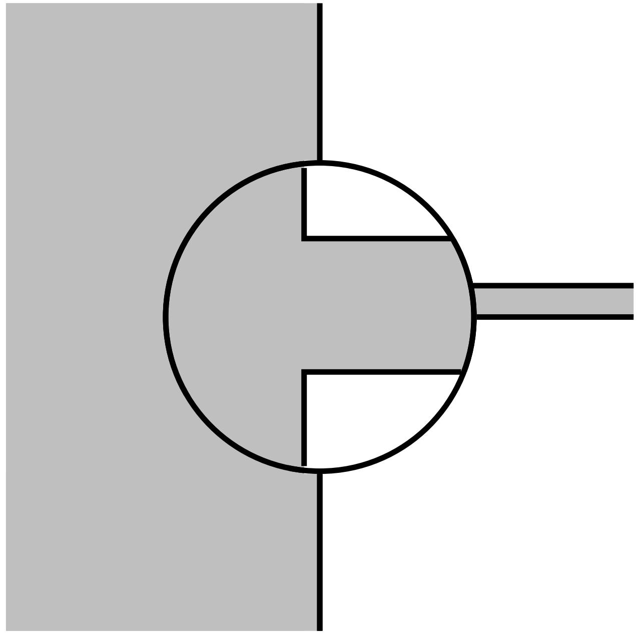
where the U_i^k satisfy the 1D Helmholtz equation:

$$\frac{d^2 U_i^k}{dx^2} + \omega^2 \, U_i^k = 0$$

Near field

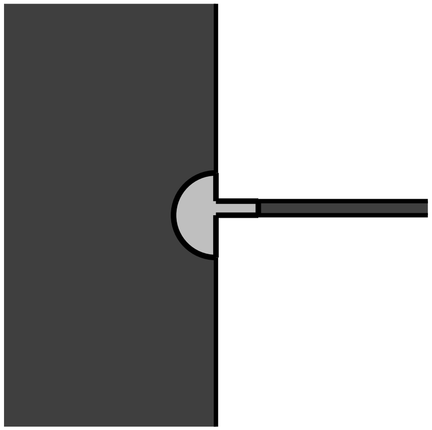


Near field



$$u^{\varepsilon}(x,y) = u_p^{\varepsilon}\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)$$

Near field



The Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda, \quad \varepsilon \ll \eta_S \ll \lambda.$

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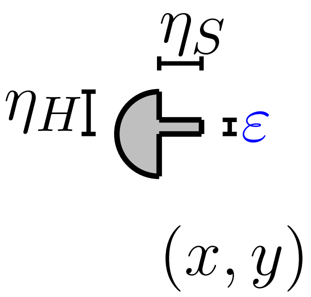
Near field



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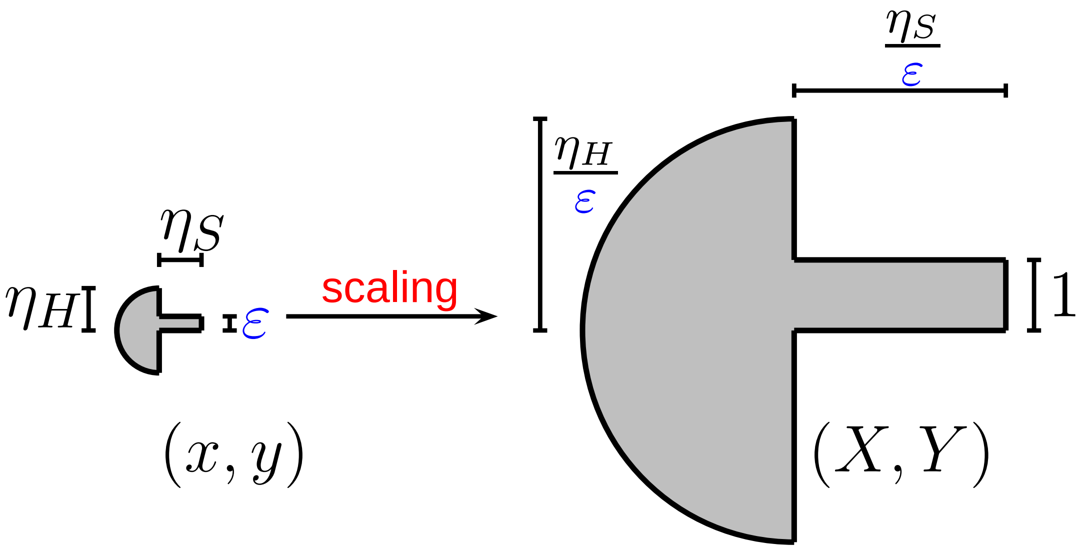

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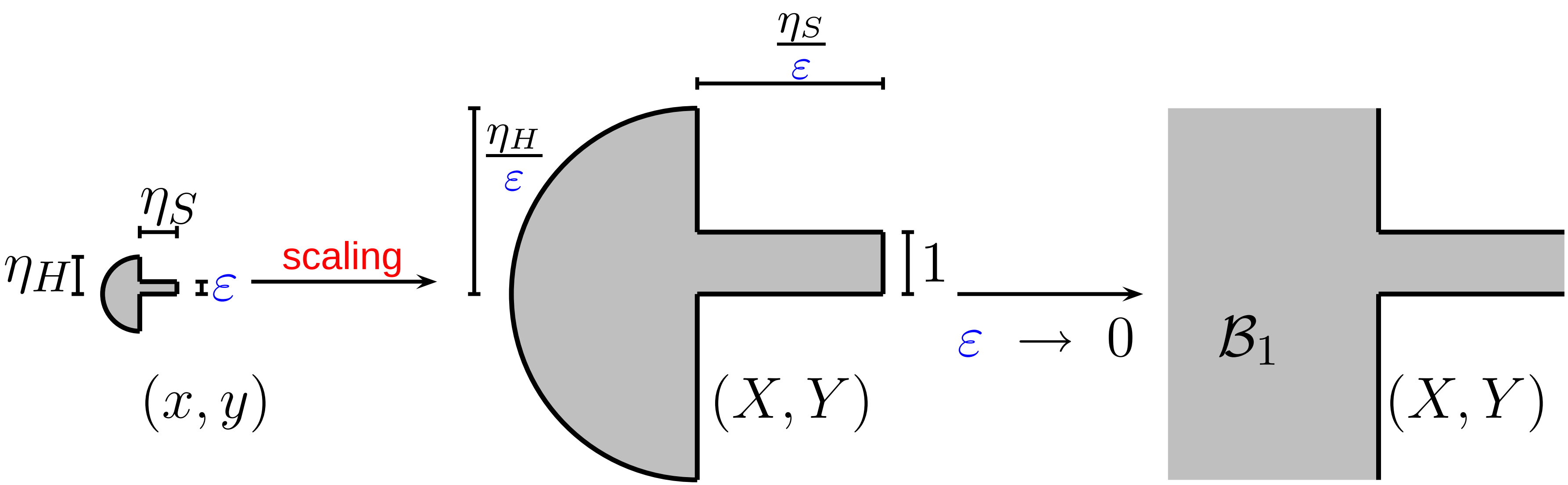

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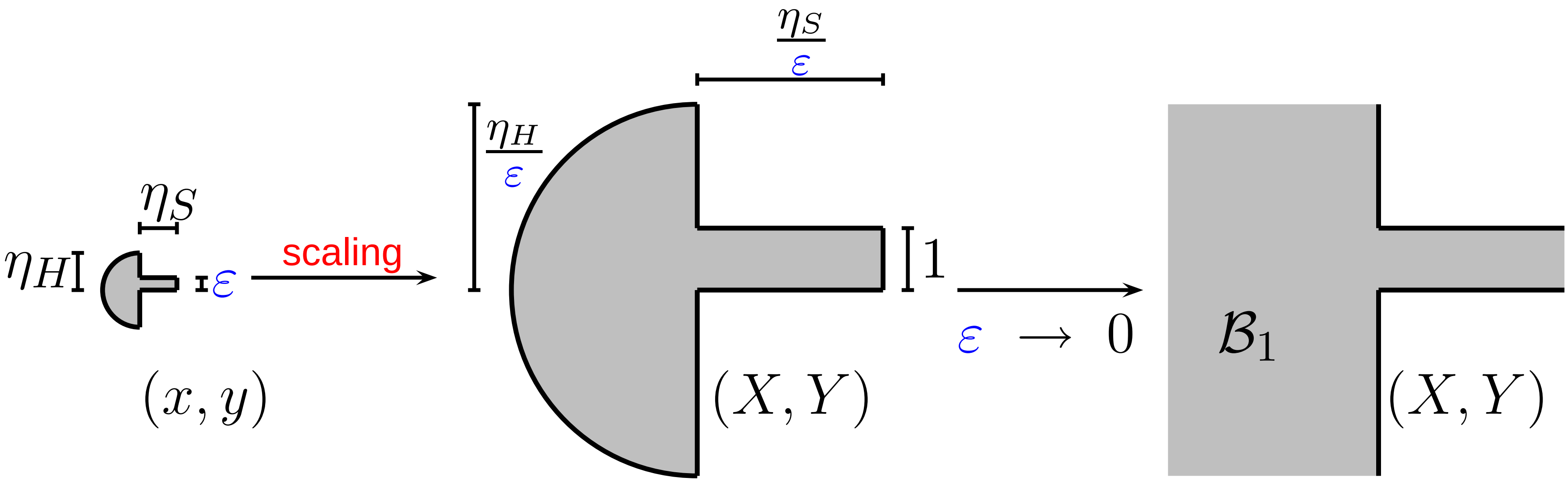
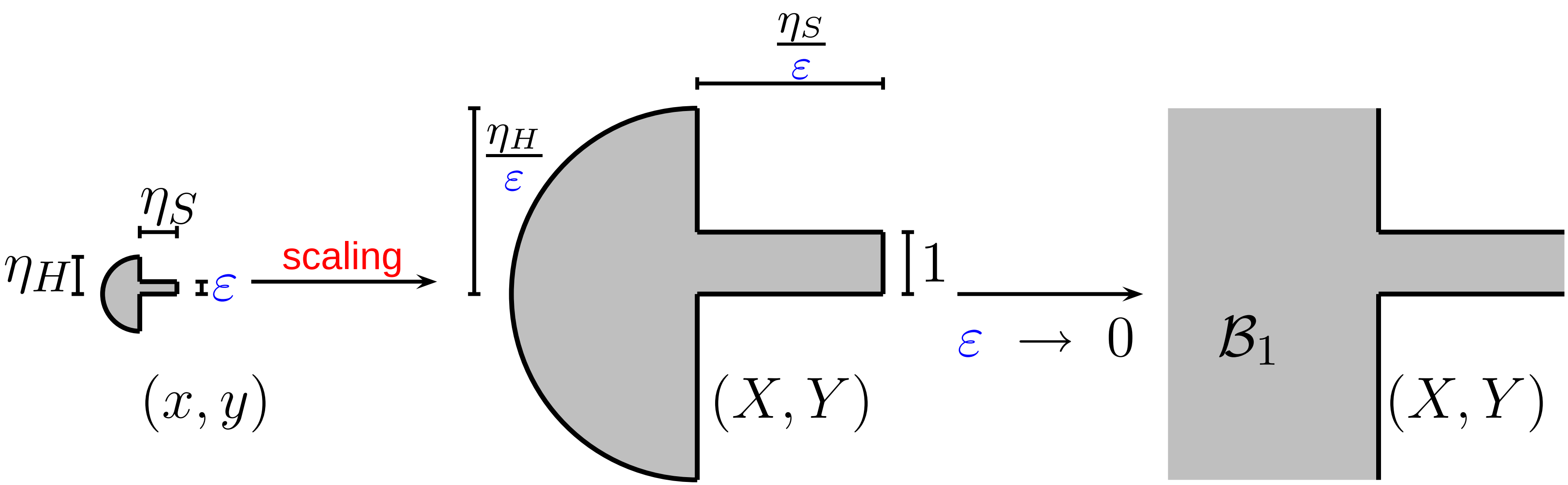


$$u^\varepsilon(\varepsilon X, \varepsilon Y) = u_p^\varepsilon(X, Y) = \sum_{i=0}^{+\infty} \sum_{k=0}^{i} \varepsilon^i (\log \varepsilon)^k \, (u_p)_i^k(X, Y) + o(\varepsilon^\infty)$$

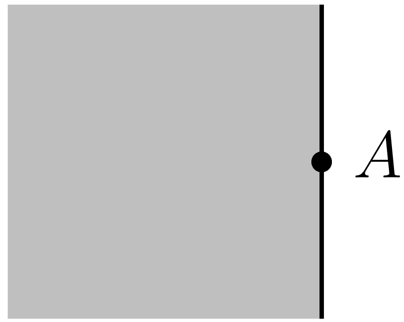
Near field



where the $(u_p)_i^k$ satisfy the (in)-homogeneous Laplace equation.

$$\begin{cases} \Delta (u_p)_i^k = 0, & \text{if } i = k \text{ or } k+1, \\ \Delta (u_p)_i^k = -\,\omega^2\,(u_p)_{i-2}^k, & \text{else.} \end{cases}$$

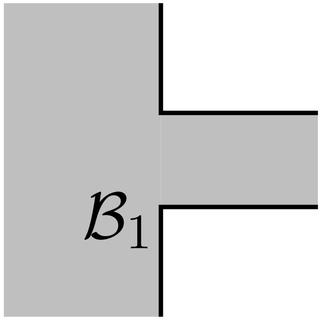
Order 0 : $\underline{u^0}$, $(u_p)_0^0$, U_0^0



Far field:

$$\begin{cases} \text{Find } u^0 \in H^1_{loc}(\Omega) \text{ such that :} & \\ -\Delta u^0 - \omega^2 u^0 = f, & \text{in } \Omega, \\ \dfrac{\partial u^0}{\partial n} = 0, & \text{on } \partial\Omega, \\ u^0 \text{ is outgoing.} & \end{cases}$$

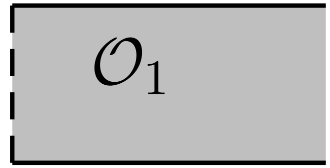
Order 0 : u^0, $\underline{(u_p)_0^0}$, U_0^0



Near field:

$$(u_p)_0^0\,(X,Y) \;=\; u^0(A), \qquad\qquad \text{in } \mathcal{B}_1.$$

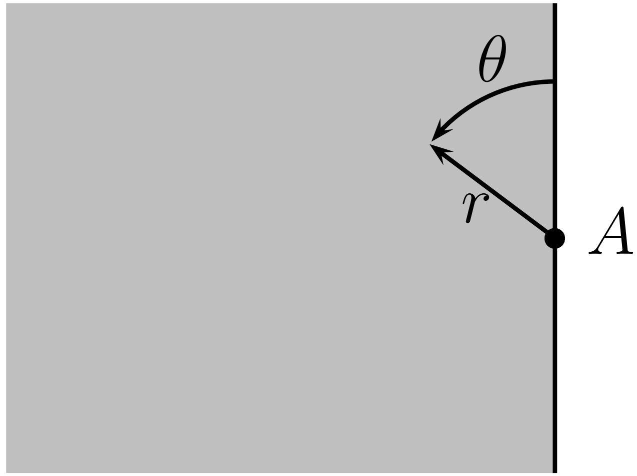
Order 0 : u^0, $(u_p)^0_0$, $\underline{U^0_0}$



Slot field:

$$U^0_0(x,Y) \;=\; u^0(A)\,\text{exp}(\mathbf{i}\omega\; x), \qquad\qquad \text{in } \mathcal{O}_1.$$

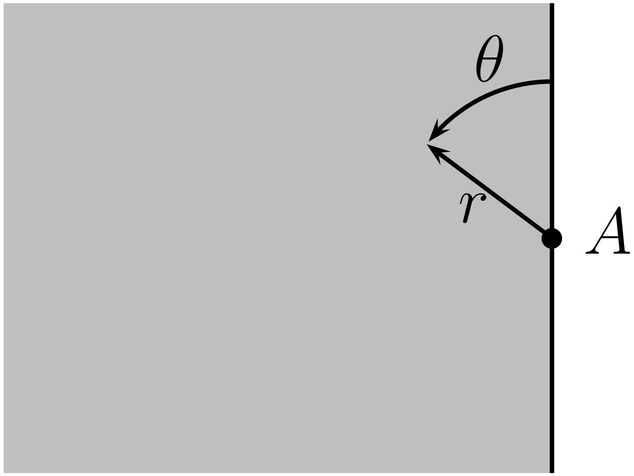
Order 1 : $\underline{u_1^0}$, $(u_p)_1^0$, $(u_p)_1^1$, U_1^0, U_1^1



Approximation of the exact Solution:

$$u^\varepsilon \simeq u^0 + \varepsilon\, u_1^0$$

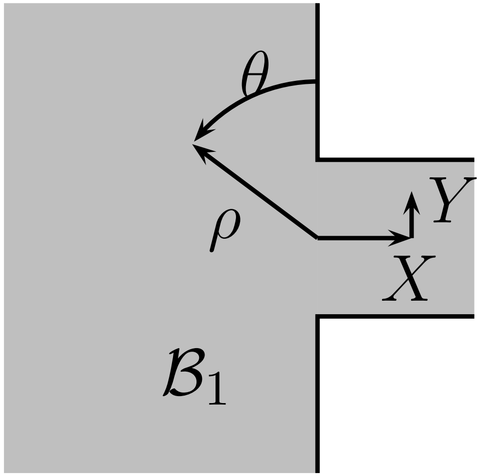
Order 1 : $\underline{u_1^0}$, $(u_p)_1^0$, $(u_p)_1^1$, U_1^0, U_1^1



explicit form of u_1^0

$$u_1^0(r,\theta) \;=\; -\frac{\omega}{2}\, u^0(A)\; H_0^{(1)}(\omega r).$$

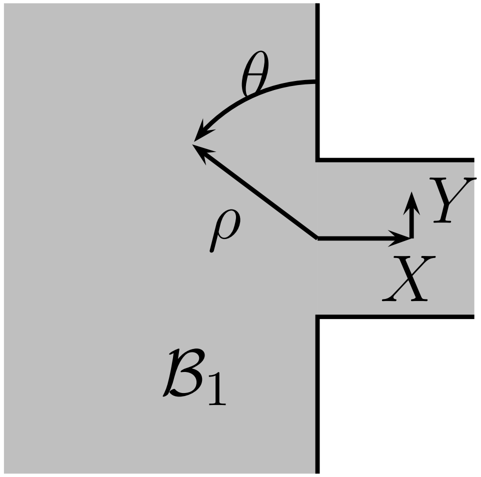
Order 1 : u_1^0, $\underline{(u_p)_1^0}$, $\underline{(u_p)_1^1}$, U_1^0, U_1^1



Approximation of the exact solution:

$$\left\{\begin{array}{l} u^\varepsilon(\varepsilon X, \varepsilon Y) \;=\; u_p^\varepsilon(X, Y), \\ u_p^\varepsilon \simeq (u_p)_0^0 \;+\; \varepsilon\;(u_p)_1^0 \;+\; \varepsilon\;\log\varepsilon\;(u_p)_1^1. \end{array}\right.$$

Order 1 : u_1^0, $\underline{(u_p)_1^0}$, $(u_p)_1^1$, U_1^0, U_1^1

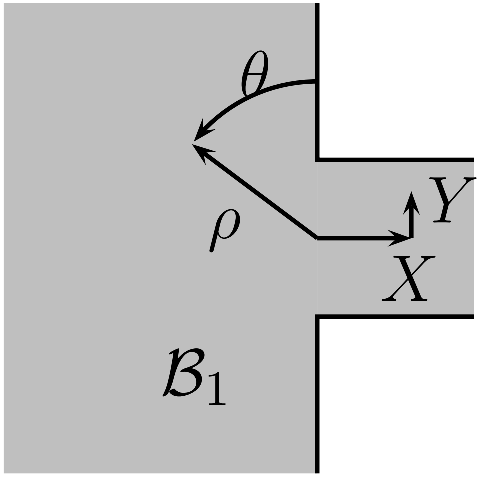


Near field:

Find $(u_p)_1^0 \in H^1_{loc}(\mathcal{B}_1)$ such that:

$$\begin{cases} \Delta (u_p)_1^0 = 0, & \text{in } \mathcal{B}_1 \\ \dfrac{\partial (u_p)_1^0}{\partial n} = 0, & \text{on } \partial\mathcal{B}_1. \end{cases}$$

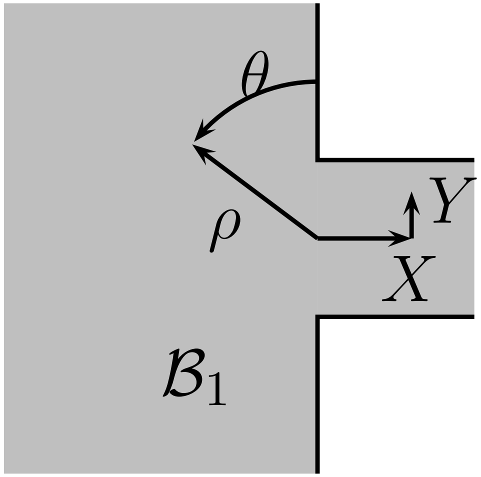
Order 1 : u_1^0, $\underline{(u_p)_1^0}$, $(u_p)_1^1$, U_1^0, U_1^1



Behavior at infinity in the half-space:

$$(u_p)_1^0(\rho,\theta) - \frac{\partial u^0}{\partial y}(A)\;\rho\;\cos\theta\;+\;\frac{\omega}{2}\;u^0(A)\;\Big[1+\frac{2\mathbf{i}}{\pi}\big(\log\rho+\gamma\big)\Big]\;=\;O(\frac{1}{\rho}).$$

Order 1 : $u_1^0, \underline{(u_p)_1^0}, (u_p)_1^1, U_1^0, U_1^1$



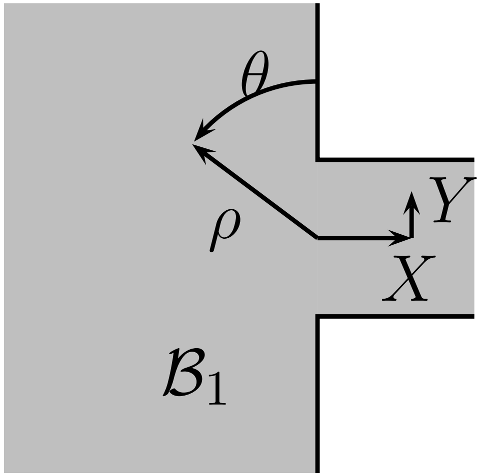
Behavior at infinity in the half-space:

$$(u_p)_1^0(\rho,\theta) - \frac{\partial u^0}{\partial y}(A)\,\rho\,\cos\theta + \frac{\omega}{2}\,u^0(A)\left[1 + \frac{2\mathbf{i}}{\pi}\big(\log\rho + \gamma\big)\right] = O(\frac{1}{\rho}).$$

Behavior at infity in the slot:

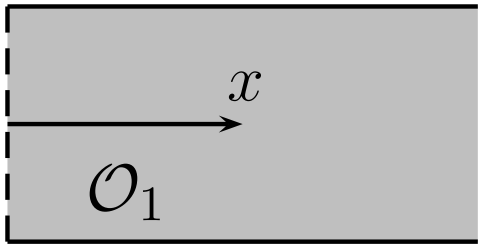
$$(u_p)_1^0\,(X,Y) - i\,\omega\,u^0(A)\,X = O(1).$$

Order 1 : u_1^0, $(u_p)_1^0$, $\underline{(u_p)_1^1}$, U_1^0, U_1^1



$$(u_p)_1^1 = -\frac{\mathbf{i}\omega}{\pi}\, u^0(A)$$

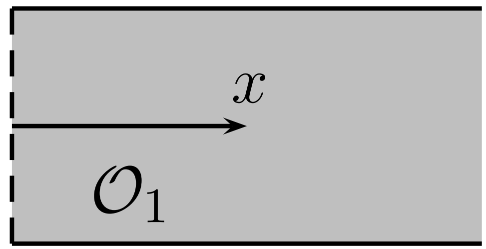
Order 1 : u_1^0, $(u_p)_1^0$, $(u_p)_1^1$, $\underline{U_1^0}$, $\underline{U_1^1}$



Approximation of the exact solution:

$$\begin{cases} u^\varepsilon(x, \varepsilon Y) \; = \; U^\varepsilon(x, Y), \\ U^\varepsilon \; \simeq \; U_0^0 \; + \; \varepsilon\, U_1^0 \; + \; \varepsilon \; \log \varepsilon \, U_1^1. \end{cases}$$

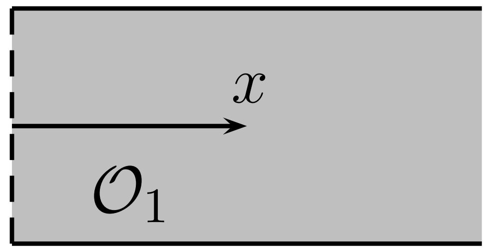
Order 1 : u_1^0, $(u_p)_1^0$, $(u_p)_1^1$, $\underline{U_1^0}$, U_1^1



The slot field:

$$U_1^0(x) \; = \; \int_0^1 \mathcal{U}_1^0(0,Y)\; dY \; \exp\left(\mathbf{i}\omega x\right),$$

Order 1 : u_1^0, $(u_p)_1^0$, $(u_p)_1^1$, U_1^0, $\underline{U_1^1}$

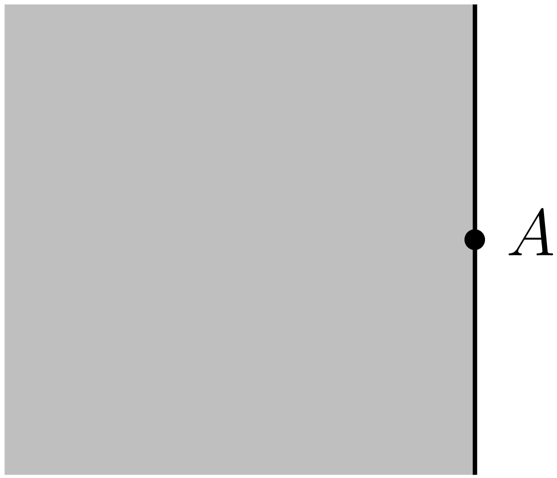


The slot field:

$$U_1^1(x) \;=\; -\frac{\mathbf{i}\omega}{\pi}\, u^0(A)\; \exp\left(\mathbf{i}\omega x\right).$$

The far fiel of order $i > 1$

- The field u_i^k are defined in the half space:



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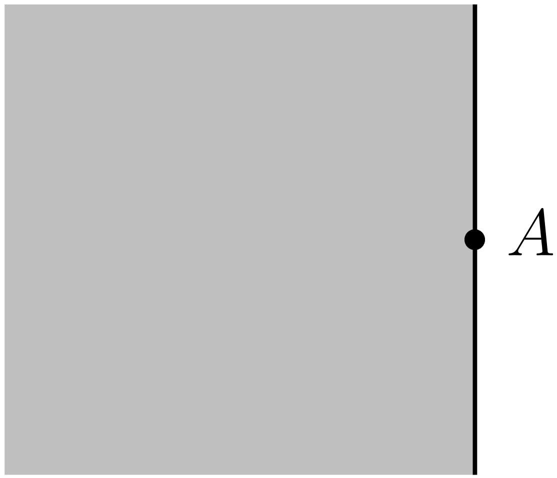

- The far fields u_i^k
  - satisfy the homogeneous Helmholtz equation
  - are singular at the neighborhood of the origin
  - are outgoing at infinity

The far fiel of order $i > 1$

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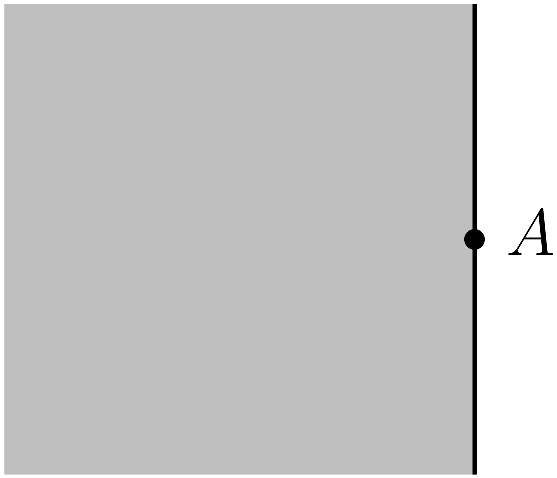

- $$u_i^k \;=\; \sum_{p=0}^{+\infty} a_p \, H_p^{(1)}(\omega r) \; \cos p\theta$$

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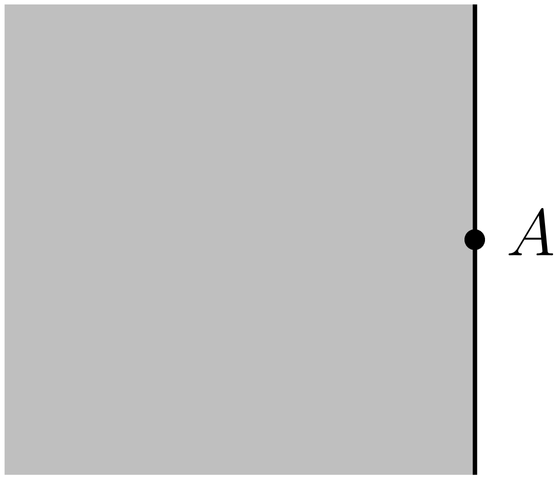

- $$u_i^k = \sum_{p=0}^{i-k-1} a_p \, H_p^{(1)}(\omega r) \; \cos p\theta$$

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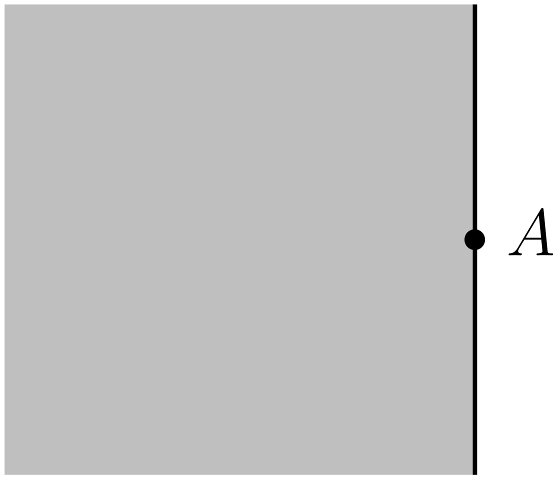

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The a_p are functions of lower order terms

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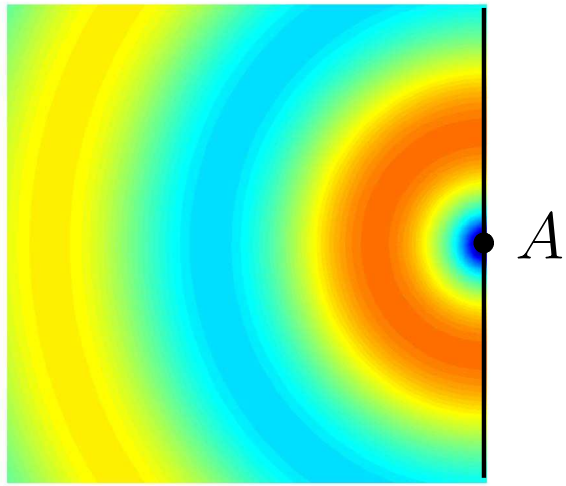
Im$(H_0^{(1)}(\omega r))$



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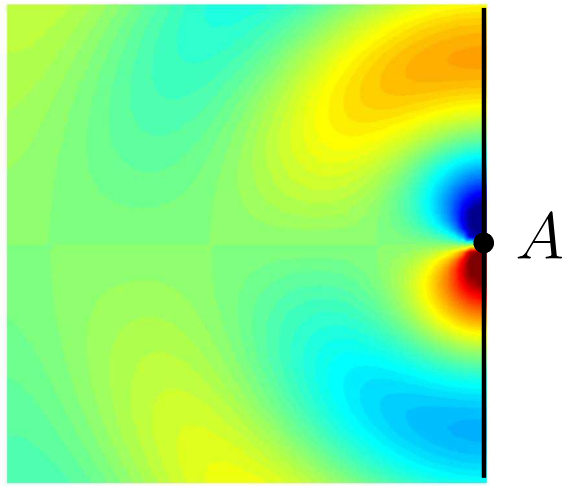

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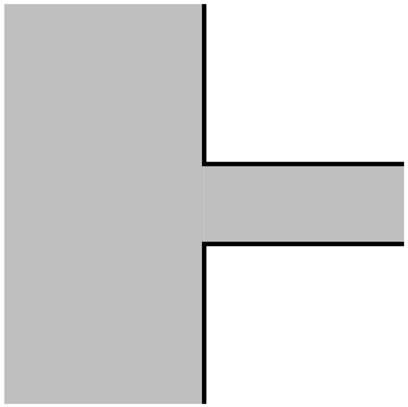
The a_p are functions of lower order terms

The near fields of order $i > 1$

- The $(u_p)_i^k(X, Y)$ are defined in the canonical domain:



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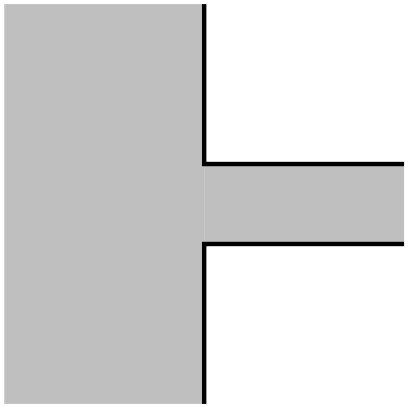

- by Laplace equation:

$$\Delta (u_p)_i^k = 0, \qquad (i = k \text{ ou } k+1),$$

$$\Delta (u_p)_i^k = -\omega^2\, (u_p)_{i-2}^k, \qquad (i \geqslant k+2),$$

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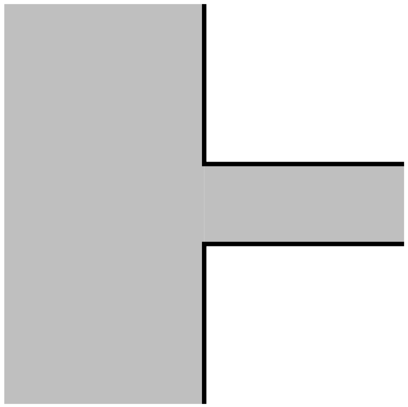

- by Laplace equation:
- by polynomial growings at infinity:
  - The growings in the half space are functions of far field of lower (or equal) order
  - The growings in the slot are functions of the slot fields of lower order

The near fields of order $i > 1$

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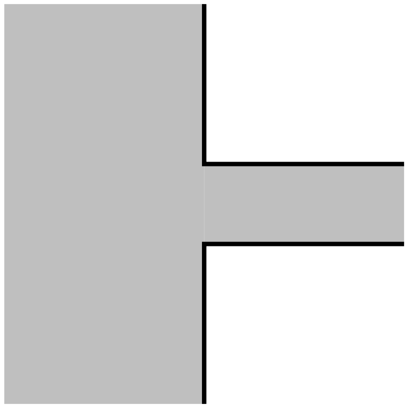
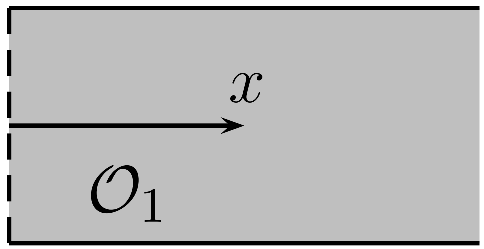


- Proof of the existence-unicity:
  - with truncature functions, we subtract the growing behavior at infinity of the $(u_p)_i^k$
  - We use the “classical” variational theory (wheighted Sobolev spaces, Leroux, Hardy,...)

The slot field of order $i > 1$

- The U_i^k are defined on the canonical domain:



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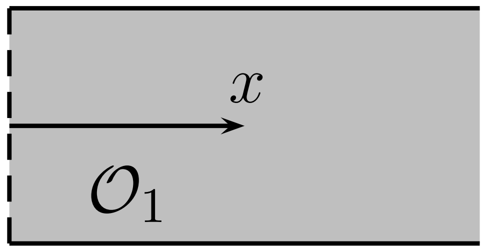

- The U_i^k does not depend on y.

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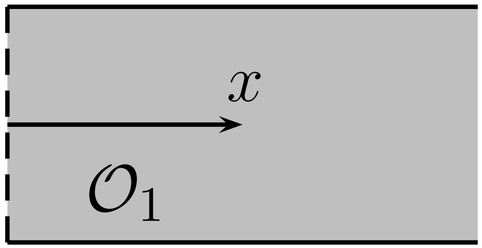

- The U_i^k does not depend on y.
- $U_i^k(x) \; = \displaystyle\int_0^1 (u_p)_i^k(0,Y)dY \;\; \exp \mathbf{i}\omega x$

Some properties

We see that:

- More $i-k$ is large more u_i^k is singular at the origin:

$$r^{-p} \text{ terms}, \quad p = 0, ..., i-k-1$$

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$$\begin{cases} \rho^p \text{ terms}, & p = 0, ..., i-k, \\ X^p \text{ termss}, & p = 0, ..., i-k, \end{cases}$$

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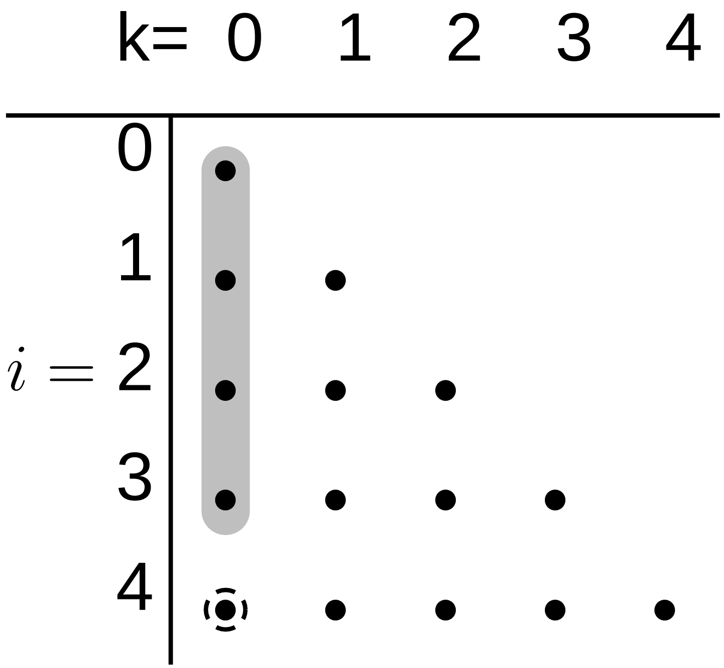
- When the order i grows, one has $O(\frac{i^2}{2})$ ($\times 3$) terms to compute...

Dependance diagram of the asymp. terms

$\varepsilon^i \, \log^k \varepsilon$



Any point corresponds to the 3 functions (u_i^k, $(u_p)_i^k$, U_i^k).

Dependance diagram of the asymp. terms

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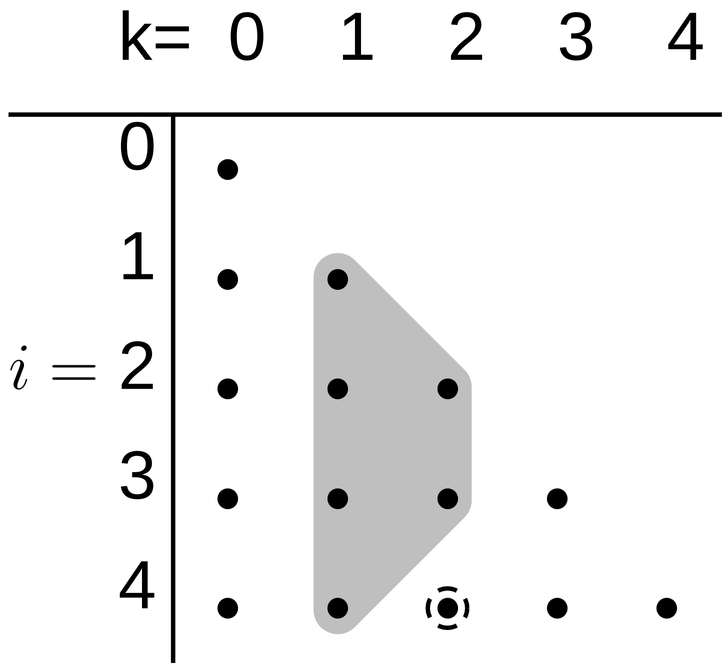
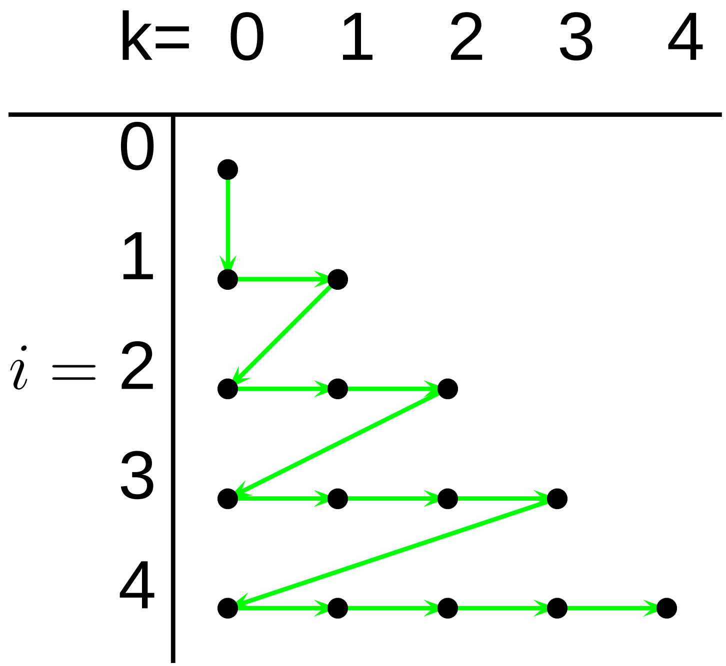


Any point corresponds to the 3 functions $(u_i^k, (u_p)_i^k, U_i^k)$.

Natural scheduling of the computations

$\varepsilon^i \ \log^k \varepsilon$



Any point corresponds to the three functions (u_i^k, $(u_p)_i^k$, U_i^k).

Devirvate the terms of the as. exp.

- We search for solutions of the form:

$$\sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \varepsilon^i (\log \varepsilon)^k \, u_i^k \qquad \text{(far field)}$$

$$\sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \varepsilon^i (\log \varepsilon)^k \, (u_p)_i^k \qquad \text{(near field)}$$

$$\sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \varepsilon^i (\log \varepsilon)^k \, U_i^k \qquad \text{(slot field)}$$

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- We inject the equations (Helmholtz, Neumann)

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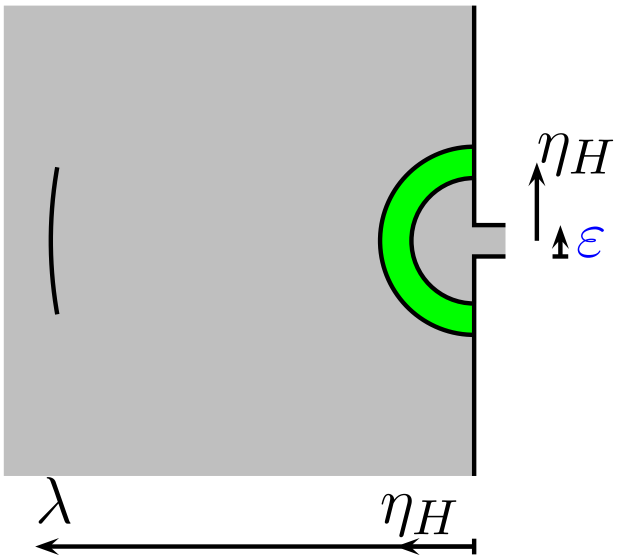
- We inject the equations (Helmholtz, Neumann)
- We obtain the coupling conditions: (the difficulty)

Far-Near coupling



In a thick zone:

$$\varepsilon \ll \eta_H \ll \lambda.$$

We write the coupling condition:

$$u^\varepsilon(\eta_H, \theta) \;=\; (u_p)^\varepsilon(\frac{\eta_H}{\varepsilon}, \theta).$$

$$\sum_{i \in \mathbb{Z}}^{+\infty} \sum_{k \in \mathbb{Z}} \varepsilon^i \,(\log \varepsilon)^k \; u_i^k(\eta_H, \theta) \;\simeq\; \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i \,(\log \varepsilon)^k (u_p)_i^k(\frac{\eta_H}{\varepsilon}, \theta)$$

$$\eta_H \to 0 \qquad\qquad \frac{\eta_H}{\varepsilon} \to +\infty$$

Far-Near coupling

$$\sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \varepsilon^i\,(\log\varepsilon)^k\,u_i^k(\eta_H,\theta) \;\simeq\; \sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \varepsilon^i\,(\log\varepsilon)^k (u_p)_i^k\Big(\frac{\eta_H}{\varepsilon},\theta\Big)$$

$$\eta_H \to 0 \qquad\qquad \frac{\eta_H}{\varepsilon} \to +\infty$$

We expand

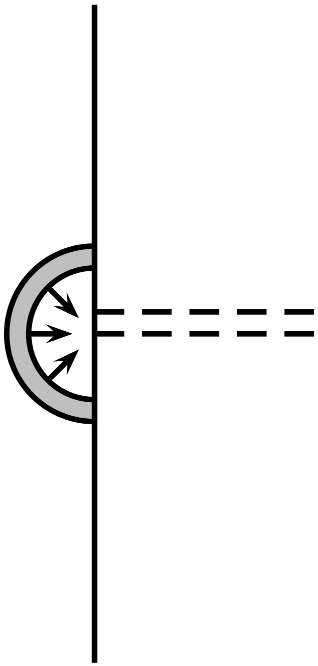
- the left serie according to η_H near 0
- The right serie according to η_H/ε tending ot infinity

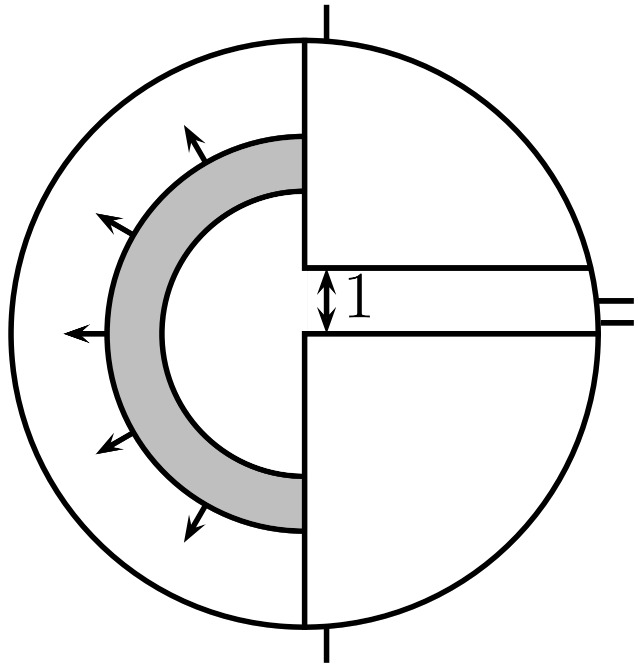
We identify all the terms of the two series.

The conclusion of the coupling

- The far field-near field coupling:

The singular behavior of the far field is coupled with the none growing behavior of the near field at infinity .

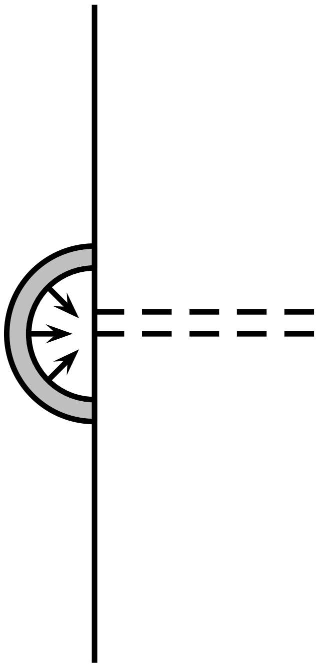


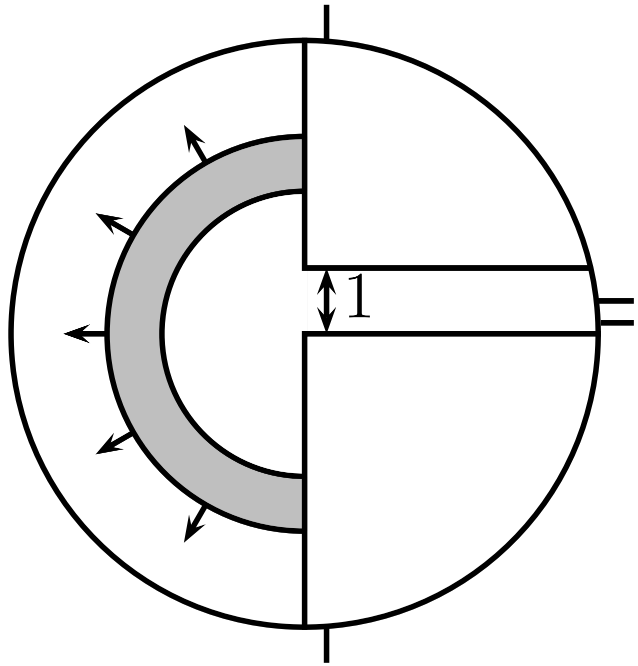


The conclusion of the coupling

- The far field-near field coupling:

The growing behavior of the near field at infinity is coupled with the none singular behavior of the far field .

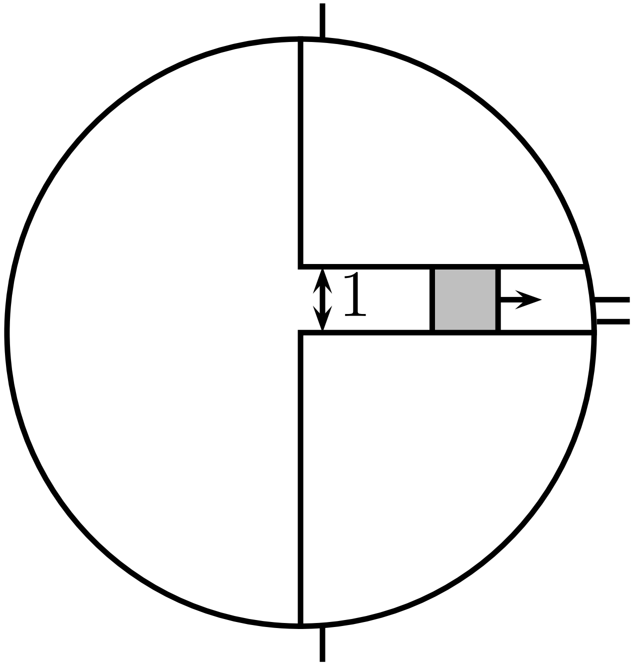


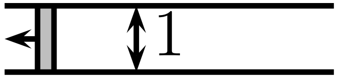


The conclusion of the coupling

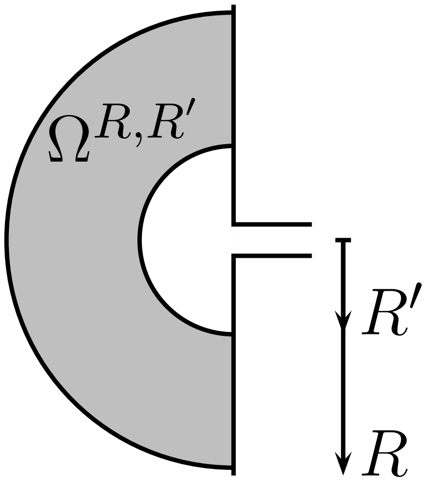
- The far field-near field coupling:
- The near field-slot field coupling:

The growing behavior of the near field is coupled with the none growing behavior of the slot field (derivative values)



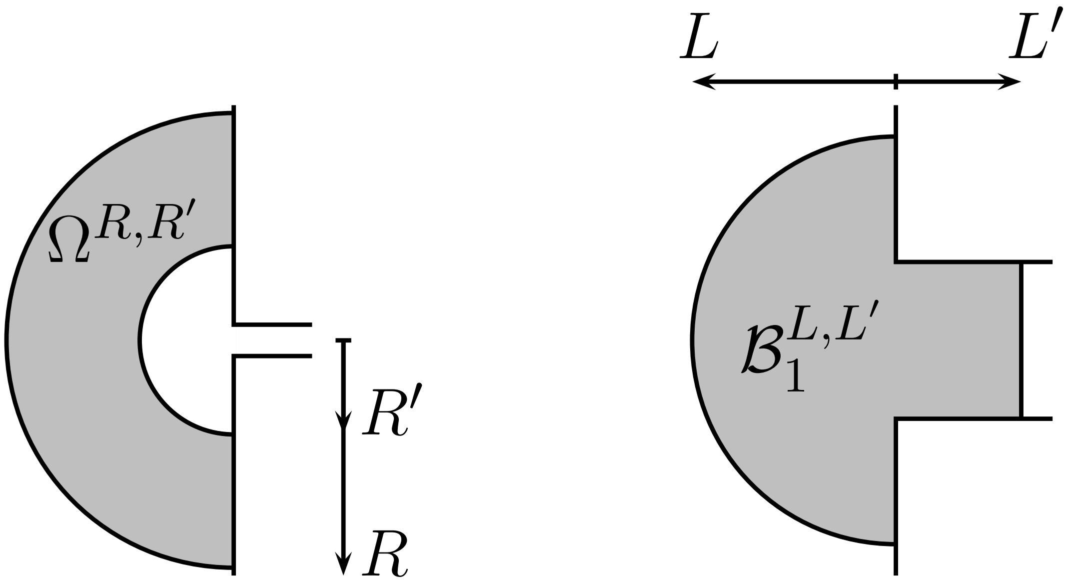


Mathematical analysis



$$\left\| u^\varepsilon - u^0 - \sum_{i=1}^{n} \sum_{k=0}^{i-1} \varepsilon^i \, (\log \varepsilon)^k \, u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \, \varepsilon^{n+1} \, (\log \varepsilon)^n \, \|f\|_{L^2(\Omega)}.$$

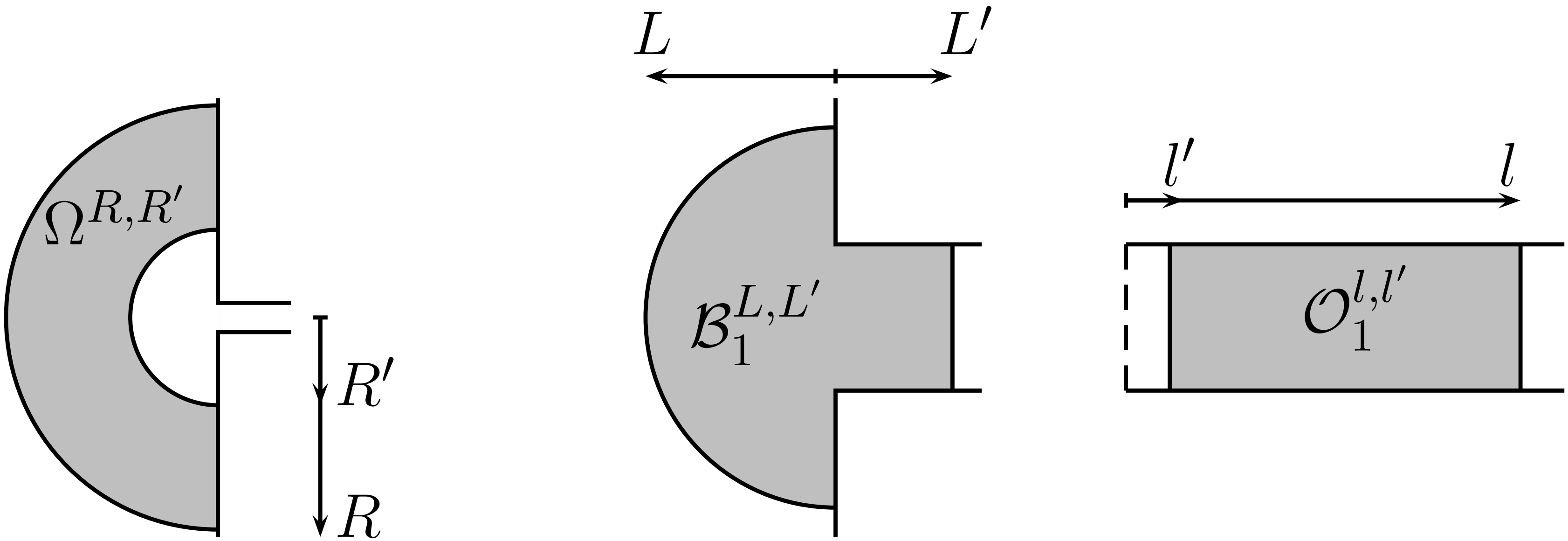
Mathematical analysis



$$\left\| u^\varepsilon - u^0 - \sum_{i=1}^{n} \sum_{k=0}^{i-1} \varepsilon^i \, (\log \varepsilon)^k \; u_i^k \right\|_{H^1(\Omega^{R,R'})} \;\leq\; C \; \varepsilon^{n+1} \, (\log \varepsilon)^n \; \|f\|_{L^2(\Omega)}.$$

$$\left\| u_p^\varepsilon - \sum_{i=0}^{n} \sum_{k=0}^{i} \varepsilon^i (\log \varepsilon)^k (u_p)_i^k \right\|_{H^1(\mathcal{B}_1^{L,L'})} \;\leq\; C \; \varepsilon^{n+1} \, (\log \varepsilon)^{n+1} \; \|f\|_{L^2(\Omega)}.$$

Mathematical analysis



$$\left\| u^\varepsilon - u^0 - \sum_{i=1}^{n} \sum_{k=0}^{i-1} \varepsilon^i \, (\log \varepsilon)^k \; u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \, \varepsilon^{n+1} \, (\log \varepsilon)^n \, \|f\|_{L^2(\Omega)}.$$

$$\left\| u_p^\varepsilon - \sum_{i=0}^{n} \sum_{k=0}^{i} \varepsilon^i (\log \varepsilon)^k (u_p)_i^k \right\|_{H^1(\mathcal{B}_1^{L,L'})} \leq C \, \varepsilon^{n+1} \, (\log \varepsilon)^{n+1} \, \|f\|_{L^2(\Omega)}.$$

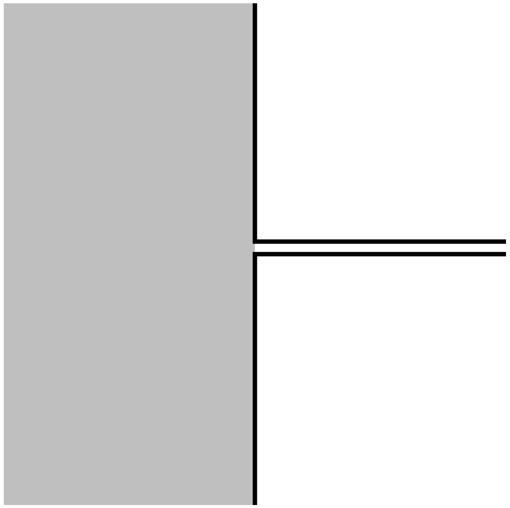
$$\left\| U^\varepsilon - \sum_{i=0}^{n} \sum_{k=0}^{i} \varepsilon^i \, (\log \varepsilon)^k \; U_i^k \right\|_{H^1(\mathcal{O}_1^{l,l'})} \leq C \, \varepsilon^{n+1} \, (\log \varepsilon)^{n+1} \, \|f\|_{L^2(\Omega)}.$$

Idea of the proof

We want to define an approximation $\widetilde{u}_n^{\varepsilon}$ of the exact solution which coincide with:

- the truncated expansion of the far field away from the slot in the half space.

$$u_n^{H,\varepsilon}(x,y) \;=\; u^0(x,y) + \sum_{i=1}^{n}\sum_{k=0}^{i-1} \varepsilon^i \,(\log \varepsilon)^k \; u_i^k(x,y)$$

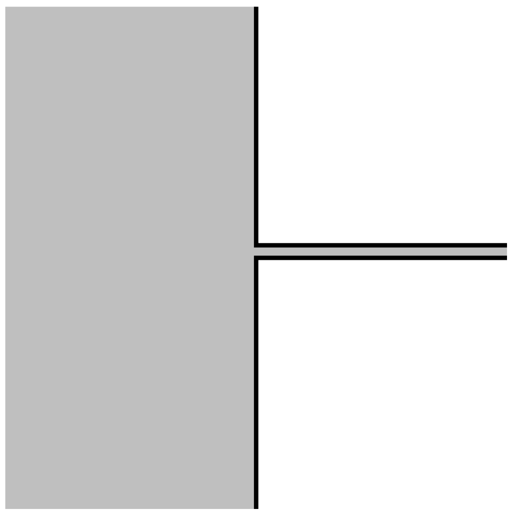


Idea of the proof

We want to define an approximation $\widetilde{u}_n^\varepsilon$ of the exact solution which coincide with:

- The truncated expansion of the near field in the neighbourhood of the end of the slot

$$u_n^{N,\varepsilon}(x,y) \;=\; \sum_{i=0}^{n}\sum_{k=0}^{i}\varepsilon^i\,(\log\varepsilon)^k\,(u_p)_i^k\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)$$

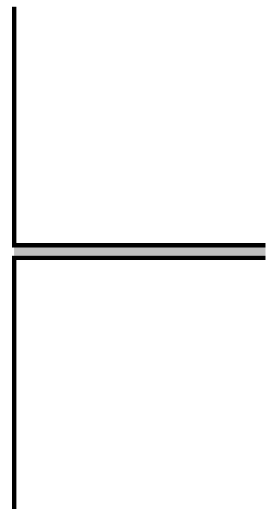


Idea of the proof

We want to define an approximation $\widetilde{u}_n^{\varepsilon}$ of the exact solution which coincide with:

- the truncated expansion of the slot field far away in the slot

$$u_n^{S,\varepsilon}(x,y) \;=\; \sum_{i=0}^{n}\sum_{k=0}^{i} \varepsilon^i \,(\log \varepsilon)^k \; U_i^k(x, \frac{y}{\varepsilon})$$



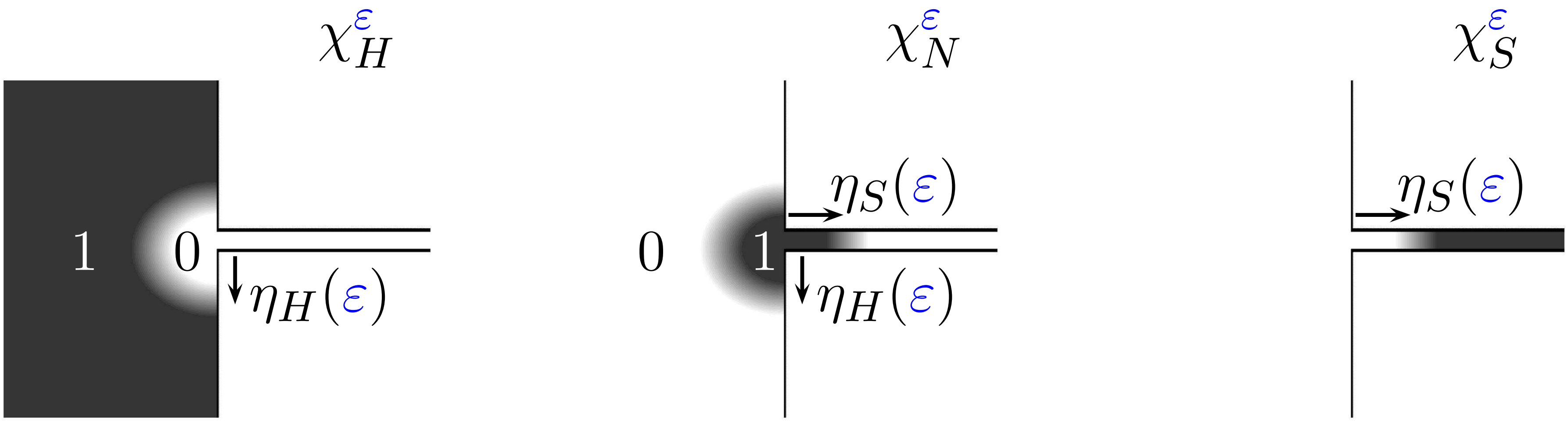
Idea of the proof

Introduce a partition of unity

$$\widetilde{u}_n^{\varepsilon}(r,\theta) \;=\; \chi_H^{\varepsilon}\, u_n^{H,\varepsilon} \;+\; \chi_N^{\varepsilon}\, u_n^{N,\varepsilon} \;+\; \chi_S^{\varepsilon}\, u_n^{S,\varepsilon}$$

with

$$\chi_H^{\varepsilon} \;+\; \chi_N^{\varepsilon} \;+\; \chi_S^{\varepsilon} \;=\; 1.$$

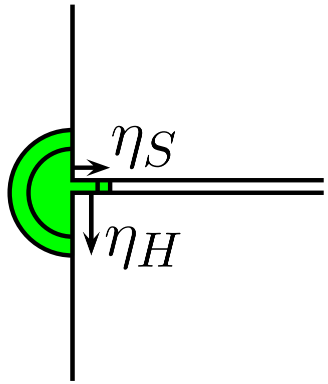


Idea of the proof

The error equation: $e_n^\varepsilon = \widetilde{u}_n^\varepsilon - u^\varepsilon$

$$\begin{cases} \Delta e_n^\varepsilon + \omega^2 e_n^\varepsilon = (\delta_N)_n^\varepsilon + (\delta_{H-N})_n^\varepsilon + (\delta_{S-N})_n^\varepsilon, & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial e_n^\varepsilon}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ e_n^\varepsilon \text{ is outgoing.} & \end{cases}$$

$(\delta_N)_n^\varepsilon$ is related to the approximation of the Helmholtz equation by the near field

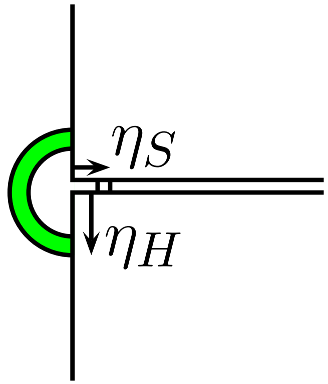


Idea of the proof

The error equation $e_n^\varepsilon = \widetilde{u}_n^\varepsilon - u^\varepsilon$

$$\begin{cases} \Delta e_n^\varepsilon + \omega^2 e_n^\varepsilon = (\delta_N)_n^\varepsilon + (\delta_{H-N})_n^\varepsilon + (\delta_{S-N})_n^\varepsilon, & \text{dans } \Omega_\varepsilon, \\ \dfrac{\partial e_n^\varepsilon}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ e_n^\varepsilon \text{ is outgoing.} & \end{cases}$$

$(\delta_{H-N})_n^\varepsilon$ is related to the matching error between the far field ans the near field

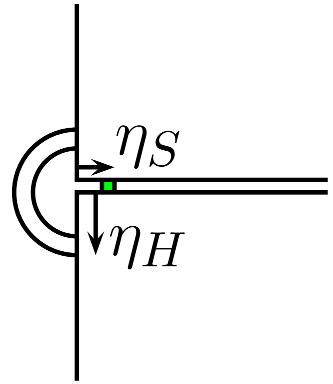


Idea of the proof

The error equation $e_n^\varepsilon = \widetilde{u}_n^\varepsilon - u^\varepsilon$

$$\begin{cases} \Delta e_n^\varepsilon + \omega^2 e_n^\varepsilon = (\delta_N)_n^\varepsilon + (\delta_{H-N})_n^\varepsilon + (\delta_{S-N})_n^\varepsilon, & \text{dans } \Omega_\varepsilon, \\ \dfrac{\partial e_n^\varepsilon}{\partial n} = 0, & \text{sur } \partial\Omega_\varepsilon, \\ e_n^\varepsilon \text{ is outgoing.} & \end{cases}$$

$(\delta_{S-N})_n^\varepsilon$ is related to the matching error between the slot field and the near champ



Idea of the proof

The error equation: $e_n^\varepsilon = \widetilde{u}_n^\varepsilon - u^\varepsilon$

$$\begin{cases} \Delta e_n^\varepsilon + \omega^2 e_n^\varepsilon = (\delta_N)_n^\varepsilon + (\delta_{H-N})_n^\varepsilon + (\delta_{S-N})_n^\varepsilon, & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial e_n^\varepsilon}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ e_n^\varepsilon \text{ os outgoing.} & \end{cases}$$

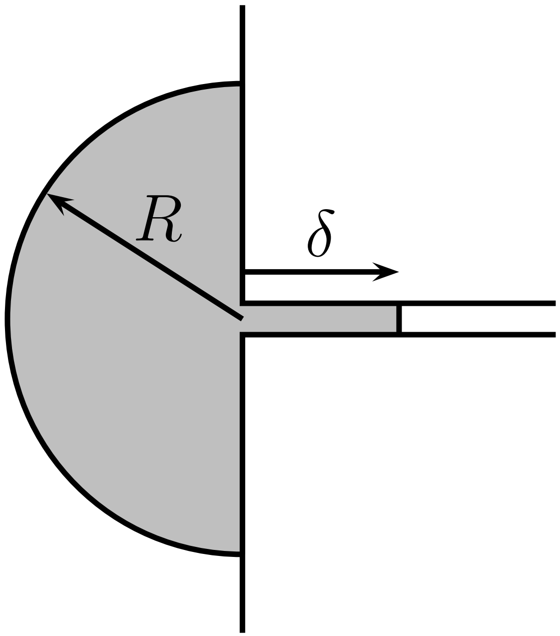
classical asymptotic techniques:

- Stability: proof by contradiction (Helmholtz)
- Consistency: A little bit more difficult (study of the singularities and of the growings by separation of variable)

Idea of the proof

Global error estimates

$$\left\{\begin{aligned} \left\| u^{\varepsilon} - \widetilde{u}_n^{\varepsilon} \right\|_{H^1(\Omega_\varepsilon^{R,\delta})} \quad \leqslant \quad & C\left[\Big(\eta_H(\varepsilon)\Big)^n + \left(\frac{\varepsilon}{\eta_H(\varepsilon)}\right)^n\right] \\ & + C\left[\Big(\eta_S(\varepsilon)\Big)^n + \left(\frac{\varepsilon}{\eta_S(\varepsilon)}\right)^n\right]. \end{aligned}\right.$$



Idea of the proof

Global error estimate

$$\begin{cases} \left\| u^{\varepsilon} - \widetilde{u}^{\varepsilon}_{n} \right\|_{H^1(\Omega_{\varepsilon}^{R,\delta})} \leqslant & C \left[\Big(\eta_H(\varepsilon) \Big)^n + \Big(\dfrac{\varepsilon}{\eta_H(\varepsilon)} \Big)^n \right] \\ & + C \left[\Big(\eta_S(\varepsilon) \Big)^n + \Big(\dfrac{\varepsilon}{\eta_S(\varepsilon)} \Big)^n \right]. \end{cases}$$

One can choose $\eta_H(\varepsilon)$ and $\eta_S(\varepsilon)$ to optimize this relation

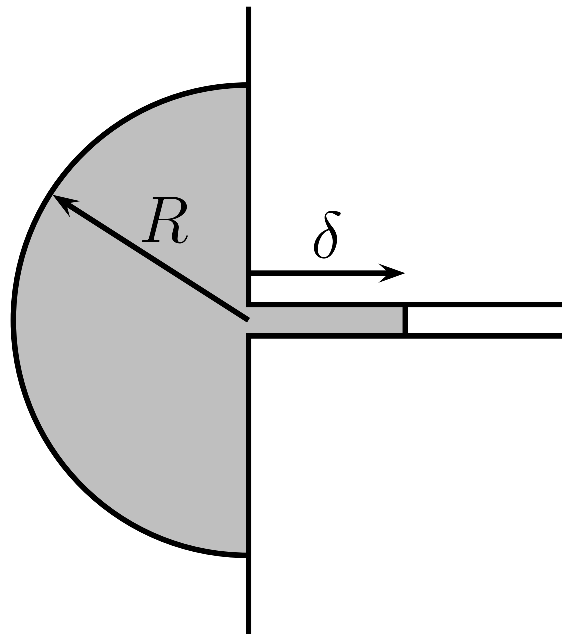
$$\eta_H(\varepsilon) = \eta_S(\varepsilon) = \sqrt{\varepsilon}$$

This leads to

$$\left\| u^{\varepsilon} - \widetilde{u}^{\varepsilon}_{n} \right\|_{H^1(\Omega_{\varepsilon}^{R,\delta})} \leqslant C \, \varepsilon^{\frac{n}{2}}$$

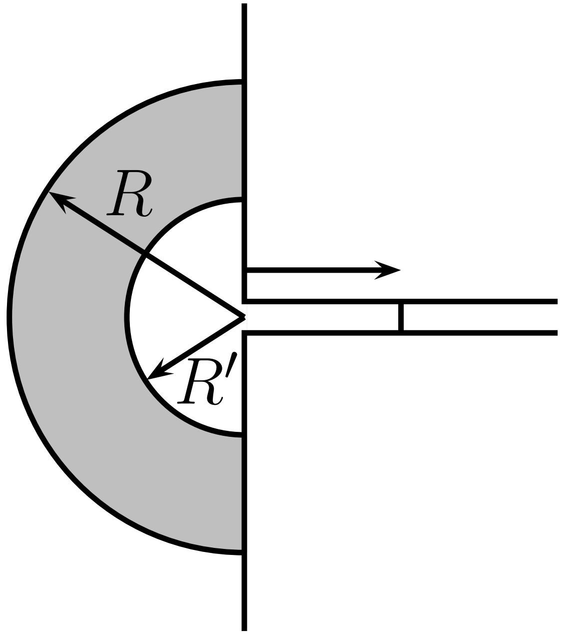
Idea of the proof

$$\left\| u^{\varepsilon} - \widetilde{u}^{\varepsilon}_{n} \right\|_{H^1(\Omega_{\varepsilon}^{R,\delta})} \leqslant C\, \varepsilon^{\frac{n}{2}}$$



Idea of the proof

$$\left\| u^{\varepsilon} - \widetilde{u}_n^{\varepsilon} \right\|_{H^1(\Omega_{\varepsilon}^{R,\delta})} \leqslant C\, \varepsilon^{\frac{n}{2}} \quad \Longrightarrow \quad \left\| u^{\varepsilon} - \widetilde{u}_n^{\varepsilon} \right\|_{H^1(\Omega^{R,R'})} \leqslant C\, \varepsilon^{\frac{n}{2}}$$



Idea of the proof

$$\left\| u^{\varepsilon} - \widetilde{u}_n^{\varepsilon} \right\|_{H^1(\Omega_{\varepsilon}^{R,\delta})} \leqslant C\,\varepsilon^{\frac{n}{2}} \quad \Longrightarrow \quad \left\| u^{\varepsilon} - \widetilde{u}_n^{\varepsilon} \right\|_{H^1(\Omega^{R,R'})} \leqslant C\,\varepsilon^{\frac{n}{2}}$$

In the far field zone:

$$\widetilde{u}_n^{\varepsilon} = u_n^{H,\varepsilon} = u^0 + \sum_{i=1}^{n} \sum_{k=0}^{i-1} \varepsilon^i \, (\log \varepsilon)^k \, u_i^k$$

Idea of the proof

$$\left\| u^{\varepsilon} - \widetilde{u}^{\varepsilon}_{n} \right\|_{H^1(\Omega^{R,\delta}_{\varepsilon})} \leqslant C\, \varepsilon^{\frac{n}{2}} \quad \Longrightarrow \quad \left\| u^{\varepsilon} - \widetilde{u}^{\varepsilon}_{n} \right\|_{H^1(\Omega^{R,R'})} \leqslant C\, \varepsilon^{\frac{n}{2}}$$

In the far field zone:

$$\widetilde{u}^{\varepsilon}_{n} = u^{H,\varepsilon}_{n} = u^{0} + \sum_{i=1}^{n} \sum_{k=0}^{i-1} \varepsilon^{i} (\log \varepsilon)^{k}\, u^{k}_{i}$$

$$\begin{cases} \left\| u^{\varepsilon} - u^{H,\varepsilon}_{3n} \right\|_{H^1(\Omega^{R,R'})} \leqslant C\, \varepsilon^{\frac{3n}{2}} \\ \left\| u^{H,\varepsilon}_{3n} - u^{H,\varepsilon}_{n} \right\|_{H^1(\Omega^{R,R'})} \leqslant C\, \varepsilon^{n+1} \log^{n} \varepsilon \end{cases}$$

Idea of the proof

$$\left\| u^{\varepsilon} - \widetilde{u}_n^{\varepsilon} \right\|_{H^1(\Omega_{\varepsilon}^{R,\delta})} \leqslant C\, \varepsilon^{\frac{n}{2}} \quad \Longrightarrow \quad \left\| u^{\varepsilon} - \widetilde{u}_n^{\varepsilon} \right\|_{H^1(\Omega^{R,R'})} \leqslant C\, \varepsilon^{\frac{n}{2}}$$

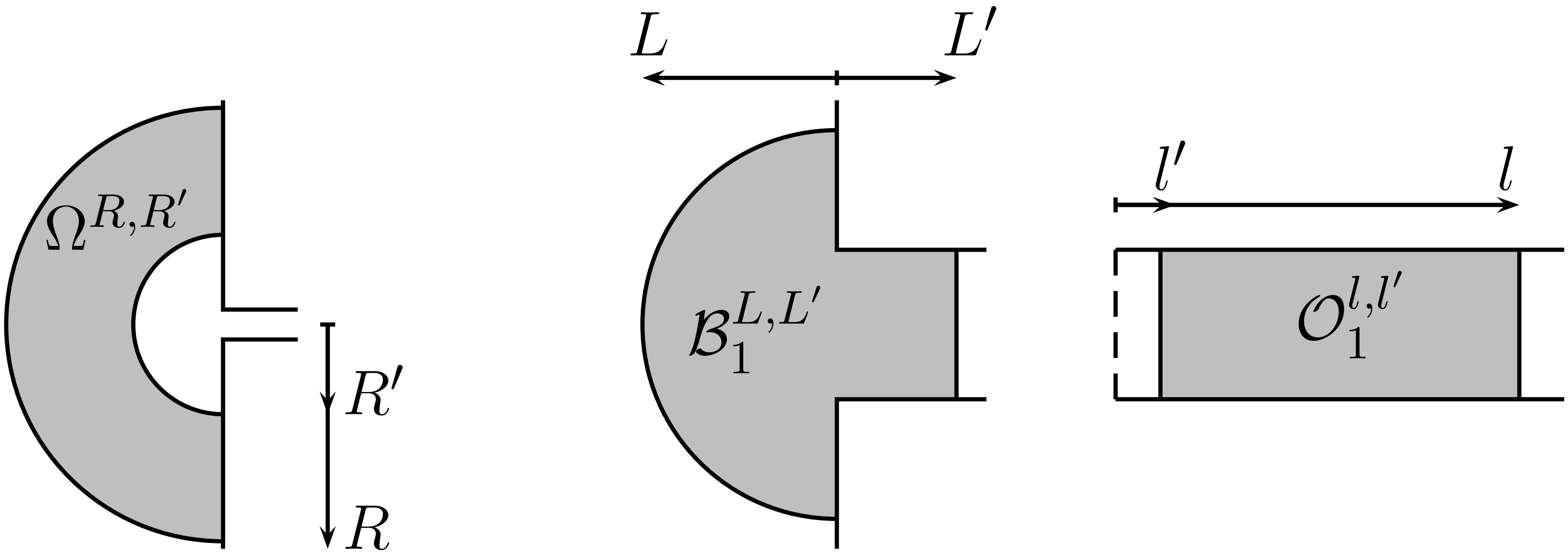
In the far field zone:

$$\widetilde{u}_n^{\varepsilon} = u_n^{H,\varepsilon} = u^0 + \sum_{i=1}^{n} \sum_{k=0}^{i-1} \varepsilon^i\, (\log \varepsilon)^k\, u_i^k$$

$$\begin{cases} \left\| u^{\varepsilon} - u_{3n}^{H,\varepsilon} \right\|_{H^1(\Omega^{R,R'})} \leqslant C\, \varepsilon^{\frac{3n}{2}} \\ \left\| u_{3n}^{H,\varepsilon} - u_n^{H,\varepsilon} \right\|_{H^1(\Omega^{R,R'})} \leqslant C\, \varepsilon^{n+1}\, \log^n \varepsilon \end{cases}$$

One can conclude using the triangular inequality.

Mathematical analysis



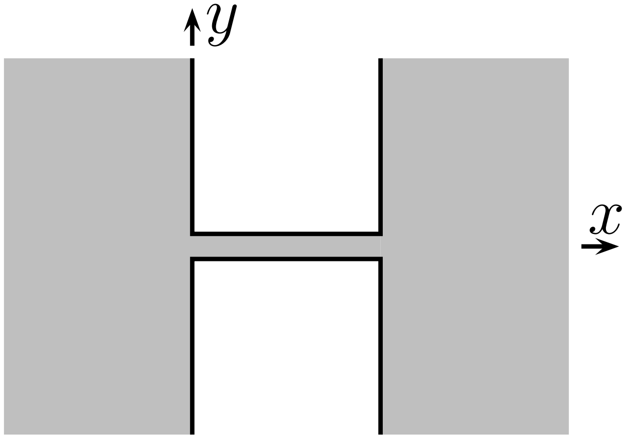
$$\left\| u^\varepsilon - u^0 - \sum_{i=1}^{n}\sum_{k=0}^{i-1} \varepsilon^i \, (\log \varepsilon)^k \, u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \, \varepsilon^{n+1} \, (\log \varepsilon)^n \, \|f\|_{L^2(\Omega)}.$$

$$\left\| u_p^\varepsilon - \sum_{i=0}^{n}\sum_{k=0}^{i} \varepsilon^i (\log \varepsilon)^k (u_p)_i^k \right\|_{H^1(\mathcal{B}_1^{L,L'})} \leq C \, \varepsilon^{n+1} \, (\log \varepsilon)^{n+1} \, \|f\|_{L^2(\Omega)}.$$

$$\left\| U^\varepsilon - \sum_{i=0}^{n}\sum_{k=0}^{i} \varepsilon^i \, (\log \varepsilon)^k \, U_i^k \right\|_{H^1(\mathcal{O}_1^{l,l'})} \leq C \, \varepsilon^{n+1} \, (\log \varepsilon)^{n+1} \, \|f\|_{L^2(\Omega)}.$$

Perspectives

1. Mathematical analysis of the finite slot (resonance phenomena)



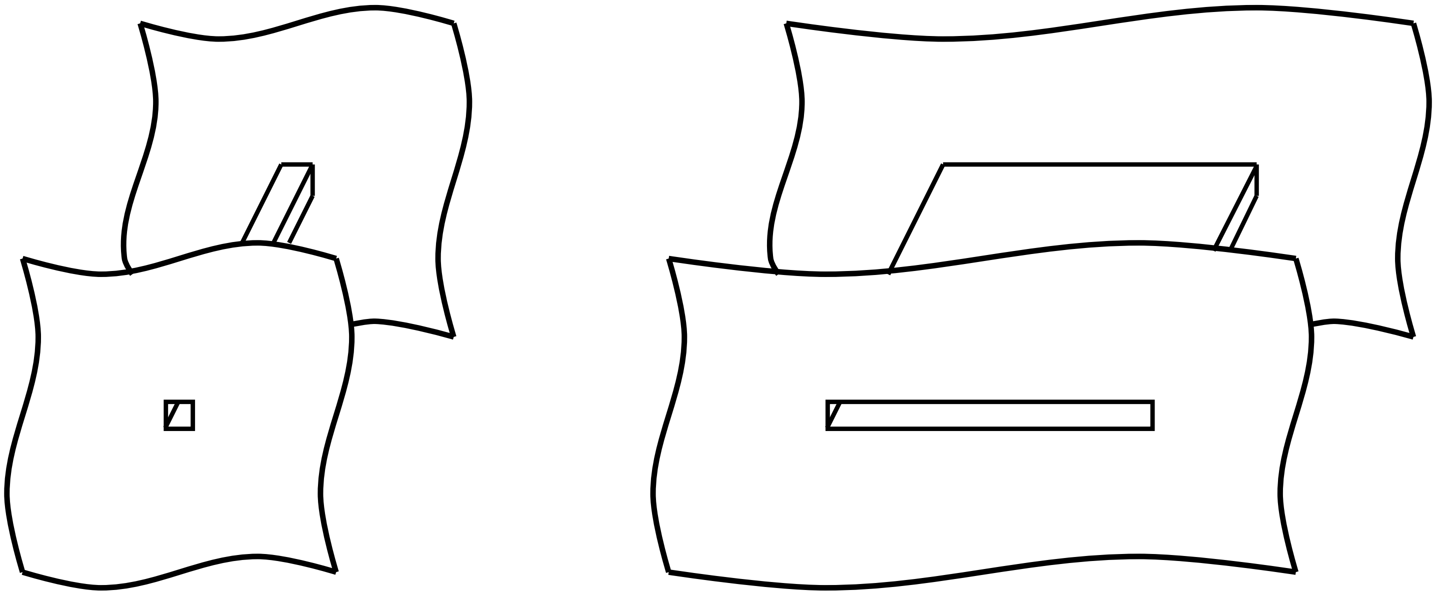
The difficulty: the stability result.

Perspectives

1. Mathematical analysis of the finite slot (resonance phenomena)
2. Comparison with the multi-scale technique

Perspectives

1. Mathematical analysis of the finite slot (resonance phenomena)
2. Comparison with the multi-scale technique
3. The 3D Maxwell equation



Perspectives

1. Mathematical analysis of the finite slot (resonance phenomena)
2. Comparison with the multi-scale technique
3. The 3D Maxwell equation
4. The time domain (evolution equation)

$$\frac{\partial^2 u}{\partial t^2} - c^2 \,\Delta u = 0.$$